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METHOD OF CONJUGATE GRADIENTS FOR OPTIMAL CONTROL PROBLEMS WITH--ETC(U)
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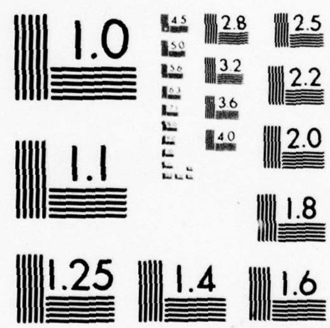

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METHOD OF CONJUGATE GRADIENTS FOR OPTIMAL CONTROL PROBLEMS WITH STATE VARIABLE CONSTRAINTS

UNIVERSITY OF CALIFORNIA, LOS ANGELES
SCHOOL OF ENGINEERING AND APPLIED SCIENCE
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July 1978

TECHNICAL REPORT AFFDL-TR-78-85
Final Report 1977-1978

AIR FORCE FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 18 AFFDL-TR-78-85	2. GOVT ACCESSION NO.: 19 TR-78-85
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): 6 METHOD OF CONJUGATE GRADIENTS FOR OPTIMAL CONTROL PROBLEMS WITH STATE VARIABLE CONSTRAINTS.	5. TYPE OF REPORT & PERIOD COVERED: 5/77-5/78
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): 10 T. S. Fong; C. T. Leondes ~~(Principal Investigator)~~	8. CONTRACT OR GRANT NUMBER(s): F 33615-77-C-3013
9. PERFORMING ORGANIZATION NAME AND ADDRESS: School of Engineering and Applied Science, University of California, Los Angeles, Los Angeles, CA 90024	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 16 2307-03-02 17 03
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Flight Dynamics Laboratory (FGL), Wright-Patterson AFB, Ohio 45433	12. REPORT DATE: 11 July, 1978
	13. NUMBER OF PAGES: 90
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): Office of Naval Research, 1030 E. Green Street, Pasadena, CA 91106, Attention Mr. Perry Beilke — PE61102F	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

12 94 p.

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

9 Final rept. May 77-May 78,

18. SUPPLEMENTARY NOTES

15 F33615-77-C-3013

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Optimal Control, Computational Methods, Flight Control

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A review of the computational method of conjugate gradients for linear and nonlinear operator equations is given with emphasis in applying this technique to state variable constraint control problems. The first and second Frechet derivatives of the performance functional are derived. The search directions generated in the iteration process for the optimal control are locally conjugate with respect to the second Frechet derivative. The convergence is along the expanding sequence of sets, the itersection of the linear spaces spanned by the search directions and the set of admissible controls. The computational → next page

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technique is applied to two state variable constraint problems, in one of which a penalty function is employed to convert the constraint problem to an unconstrained one in addition to the approach considering the constraints directly. For this same problem the method of steepest descent also is studied, and comparison of the results obtained is made and discussed.

PREFACE

The multiple arc trajectory optimization is one which constantly confronts Air Force flight vehicle systems, both in air-to-air and air-to ground operations. The most difficult of problems here includes both bounds on state and control, and yet it cannot be avoided because this is, in fact, the situation in Air Force flight vechiles. This report appears to represent one of the most important pieces of work presenting results of importance both for flight control because of the greatly effecient algorithms developed in this report.

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SECTION I
INTRODUCTION

The past decade has seen considerable progress in techniques for optimization of nonlinear dynamical systems. The development of large digital computers coupled with the interest in optimal control theory, particularly in optimizing of spacecraft trajectories, has inspired a large volume of literature devoted to both the mathematical theory of optimal processes and the methods for obtaining solutions to these problems. Nevertheless, from the computational standpoint the class of control problems with constraint state variables has scarcely been considered, although these types of problems often occur in engineering practice. For example, the velocity of a vehicle may be limited by structure breakdown or a motor may be overloaded to prevent safety and reliability of operation. Bryson, Denham and Dreyfus [1], [2]* and Starr [3] have treated this class of problems using the steepest descent technique and a suitable combination of various non-gradient techniques, respectively. Others [2], [4] have reduced the constraint problem to unconstrained status by introducing the penalty function in place of the constraints on the state variables.

The method of steepest descent is excellent for finding an approximate solution quickly, but it often exhibits very slow convergence, whereas other techniques frequently face the problem of computational stability in the solution of the two-point boundary value problem. It is hoped that the method of conjugate gradients would

*W.F. Denham, and A.E. Bryson, Jr.,"Optimal Programming Problems with Inequality Constraints II: Solution by Steepest Descent," AIAA Journal, Vol. 2, No. 1, pp.25-34, January, 1964.

offer an improved and more efficient computational method, which is the objective of this study.

For completeness, some basic concepts from functional analysis to be used in the sequel are given in Section II, and a review of the conjugate gradient method of Hestenes and Stiefel for linear operator equations and the extension to nonlinear operator equations are given in Section III with emphasis toward control applications. Section IV discusses the class of control problems to be considered. The computational aspect of the state variable constraint control problems is presented in Section V. The application of the method of conjugate gradients to this class of optimal control problems is discussed. An algorithm is given showing the construction of the sequence of control functions that extremize a given performance functional. Sections VI and VII consider two practical engineering applications: 1) a minimum time problem in two dimensions with the constraint on the state variables being a circle, and 2) a re-entry vehicle problem with altitude constraint for which the range is to be maximized. A comparison of the rate of convergence with the method of steepest descent is given in the first problem. The results showed that the method of conjugate gradients provided a higher rate of convergence, but not as rapid as for the cases without state variable constraint.

SECTION II

BASIC CONCEPTS FROM FUNCTIONAL ANALYSIS

Some definitions and fundamental theorems from analysis which will be used in the following discussion are given below. Let V be a real Hilbert space with inner product denoted by $\langle\cdot,\cdot\rangle$. u, h are elements in V. A function $\psi(h)$ is written $o(||h||)$ if $\frac{|\psi(h)|}{||h||} \to 0$ as $||h|| \to 0$, and a function $\psi(h)$ is written $O(||h||)$ if $|\psi(h)| < N||h||$ as $||h|| \to 0$ where N is a positive constant and $||h|| = \langle h,h\rangle^{1/2}$.

Definition 2.1. If there exists a continuous linear functional G(u) on V such that

$$|E(u+h) - E(u) - G(u)h| = o(||h||) \tag{2.1}$$

as $||h|| \to 0$, then the linear functional G(u) is called the Frechet derivative of E at u and G(u)h is called the Frechet differential of E at u with increment h.

The higher derivatives are defined in a similar manner. Denote the conjugate space of V by V^*, the space of all linear functionals on V. Denote the norm on V^* by $||\cdot||^*$.

Definition 2.2. If there exists a continuous linear operator F(u) from V into V^* such that

$$||G(u+h) - G(u) - F(u)h||^* = o(||h||) \tag{2.2}$$

as $||h|| \to 0$, then the operator F(u) is called the second Frechet derivative of the functional E, and E is said to be twice differentiable. F(u)h is called the second Frechet differential.

Definition 2.3. If the limit $\frac{E(u+\theta h) - E(u)}{\theta}$ as $\theta \to 0$ exists, and let

$$\delta E(u,h) = \lim_{\theta \to 0} \frac{E(u+\theta h) - E(u)}{\theta} \tag{2.3}$$

then $\delta E(u,h)$ is called the Gateaux differential of E at u with increment h. Similarly for the second Gateaux differential at u with increments h and k,

$$\Delta E(u,h,k) = \lim_{\lambda \to 0} \frac{\delta E(u+\lambda k,h) - \delta E(u,h)}{\lambda} \tag{2.4}$$

provided the limit exists.

From the above definitions it can be seen that if the Frechet differential exists at u, then the Gateaux differential also exists, and the two differentials are equal. Although the converse may not necessarily be true, the sufficient conditions are provided by the following theorem. The proofs of the following four theorems may be found in books on functional analysis such as references [5], [6].

<u>Theorem 2.1</u>. If $\delta E(u,h)$ exists in $||u-u_o|| \le \mu$, $\mu > 0$, and if it is uniformly continuous in u and continuous in h, then the Frechet differential exists and $G(u_o)h = \delta E(u_o,h)$.

From the viewpoint of studying extremal points in function space, the concept of Frechet differential is essential, but from the computational standpoint the Frechet derivatives are often obtained through Equations (2.3) and (2.4) whenever the conditions stated in Theorem 2.1 are fulfilled.

Since $G(u)$ is a continuous linear functional, by the Riesz representation theorem there exists an element $\nabla E(u)$ in V^* such that

$$G(u)h = \langle \nabla E(u), h \rangle$$

for every $h \in V$. $\nabla E(u)$ is called the gradient of E at u. Because $F(u)h$ is a continuous linear functional on V, then

$$[F(u)h]h = \langle H_E(u)h, h\rangle$$

where $H_E(u)$ is a continuous linear operator on V. $H_E(u)$ is called the Hessian of E at u. If h is a unit vector, $\langle \nabla E(u), h\rangle$ may be regarded as a directional derivative of E in the direction of h.

Theorem 2.2. Suppose that the functional E on V has a relative extremum at u^*. It is necessary that $\nabla E(u^*) = 0$, i.e., $G(u^*)h = 0$ for all h in V.

Theorem 2.3. Suppose that the functional E on V has a relative extremum at u^* subject to constraints $\phi_j(u) = 0$, $j=1,2,\ldots,n$. Suppose that $\nabla\phi_j(u)$ exist and that they are linearly independent. If $\nabla E(u^*) \neq 0$, it is necessary that there exist unique real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all zero such that

$$\nabla E(u^*) = \sum_{1}^{n} \lambda_j \nabla\phi_j(u^*) \tag{2.5}$$

Theorem 2.4. If the Gateaux differential $\delta E(u,h)$ of a functional E exists at each point of some convex set $D \subset V$, then for any u and u+h in D,

$$E(u+h) - E(u) = \delta E(u+th, h) \tag{2.6}$$

for some t in $[0,1]$, and similar expression holds when E is an operator.

Definition 2.4. The operator F mapping V into V is called continuous at $u_o \in V$, if for any sequence $\{u_n\}$ which converges to u_o, i.e., $\lim_{n \to \infty} ||u_n - u_o|| = 0$, the sequence $\{F(u_n)\}$ converges to $F(u_o)$,

i.e., $\lim_{n \to \infty} ||F(u_n) - F(u_o)|| = 0$.

SECTION III

METHOD OF CONJUGATE GRADIENTS

3.1 Historical Survey

The method of conjugate gradients was originally developed for solving linear systems of algebraic equations independently by Hestenes and Stifel [7], [8] in 1952. Hayes [9] extended the method in 1954 to solve linear operator problems on Hilbert space. Antosiewicz and Rheinbodlt [10] in 1962 gave further consideration to the rate of convergence of the method. Fletcher and Reeves [11] in 1964 applied the conjugate gradient technique in minimizing positive definite quadratic functionals in finite dimensional space. Daniel [12] in 1965 gave an improved estimate of the rate of convergence and discussed the applicability of the conjugate gradient method to nonlinear operator equations. In the area of application of this technique to optimal control, Lasdon, Mitter and Warren [13], and Sinnott and Luenberger [14] have treated unconstrained problems with considerable success.

3.2 Linear Theory

(a) Sequence of Expanding Subspaces

Let A be a postive defintie, self-adjoint, continuous linear operator with domain V, a real Hilbert space, and range $R \subset V$. Then there exists a real number m such that (u, Au) m (u,u) for every u in V, and A has a continuous inverse A^{-1} whose domain is R and range V. The linear equation

$$Au = k \tag{3.1}$$

has a unique solution $h = A^{-1}k$ for any given $k \in R^*$. Suppose u is an estimate of h. h-u will be referred to as the error, $r = k-Au$ will be called the residual of u as an estimate of h, and

$$E(u) = \langle h-u, A(h-u) \rangle \tag{3.2}$$

is called the error functional. Since A is positive definite the problem of solving Equation (3.1) may be treated from the variational setting by minimizing the error functional in (3.2).

Finding the solution to Equation (3.1) by minimizing the error functional E in Equation (3.2) using iterative procedures often involves a sequence of expanding subspaces, i.e., a sequence of closed linear subspaces B_n of V such that

$$B_n \subset B_{n+1}$$

$$\overline{\bigcup_1^{\infty} B_n} = V .$$

The iterative procedure is the basis of the conjugate gradient method. The following three theorems illuminate the underlying philosophy of the iterative procedure.

Theorem 3.1. Let B be a linear subspace of V and u_o in V. Then the functional E in (3.2) satisfies $E(u_o) \leq E(u_o+y)$ for every y in B if and only if $\nabla E(u_o)$ or the residual at u_o is orthogonal to B. In particular, $E(u_o) \leq E(u)$ for every u in V if and only if $u_o = h$, the solution of Equation (3.2).

---

* The assumption that A is self-adjoint is not essential since from the theoretical point of view the equations Ax=k and Bx=b, where B=A*A and b=A*k, A* is the adjoint of A, are equivalent.

Proof: Let y be any non-zero element in B. Then

$$E(u_o+y) - E(u_o) = \langle y, Ay\rangle + \langle \nabla E(u_o), y\rangle .$$

Hence if $\nabla E(u_o)$ is orthogonal to B, then

$$E(u_o+y) - E(u_o) = \langle y, Ay\rangle > 0 ,$$

or $E(u_o) \leq E(u_o+y)$ for every y in B. If for every y in B, then for any real t

$$E(u_o+ty) = E(u_o) + t\langle \nabla E(u_o), y\rangle + t^2\langle y, Ay\rangle$$

which implies that $t\langle \nabla E(u_o), y\rangle + t^2\langle y, Ay\rangle \geq 0$. This expression can be true for sufficiently small t only if $\langle \nabla E(u_o), y\rangle = 0$ or $\nabla E(u_o)$ is orthogonal to B. Finally, if u_o minimizes E on V, $\nabla E(u_o) = 2(Au_o-k)$ must be orthogonal to V, and hence must be zero. But A is positive definite, therefore $u_o = h$.

It is interesting to consider a geometric interpretation of the statement of this theorem. $E(u) =$ constant defines a family of ellipsoids about h, and the gradient of E at u is orthogonal to the ellipsoid through u. The linear subspace B is a hyperplane through the origin, and u_o+B is a hyperplane through u_o. Suppose it intersects the ellipsoid $E(u) = E(u_o) = M$. Then there is a region on the hyperplane within the ellipsoid so that $E(u) \leq M$ unless u_o+B is tangent to the ellipsoid through u_o or $\nabla E(u_o)$ is orthogonal to subspace u_o+B (see Figure 3.1).

In view of the positive definite and continuity properties of E and Theorem 3.1, we have the following conclusion.

__Theorem 3.2__. Let B be a closed subspace of V. There exists a unique u_o in B that minimizes $E(u)$ on B, and $\langle \nabla E(u_o), y\rangle = 0$ for every $y \in B$.

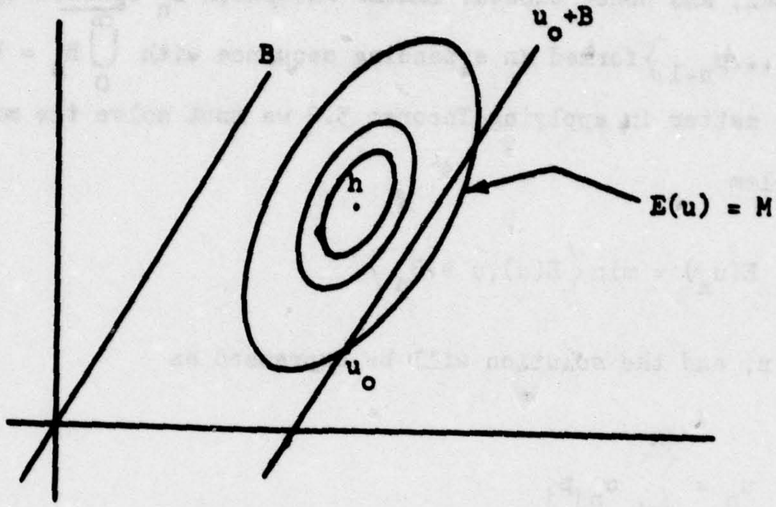


Figure 3.1. A geometric interpretation of Theorem 3.1

<u>Theorem 3.3</u>. Let $\{B_n\}$ be an expanding sequence of closed subspaces of V, and $V = \bigcup_0^\infty B_n$. Let $\{u_n\}$ be the sequence of points such that $u_n \in B_n$ and $E(u_n) = \min\{E(u),\ u \in B_n\}$. Then $u_n \to h$ as $n \to \infty$.

<u>Proof</u>: Since $\{B_n\}$ is an expanding sequence, $\{E(u_n)\}$ is a decreasing sequence, and there exists a real λ so that $E(u_n) = \lambda_n \to \lambda$ as $n \to \infty$. $E(u_n+ty) \geq \lambda$ for any t and $y \in V$; this implies that

$$\langle y, A(h-u_n)\rangle^2 \leq (\lambda_n-\lambda)\langle y, Ay\rangle ,$$

so that for $y = u_n-u_m$, $m \neq n$

$$[E(u_m)-\lambda] + [E(u_n)-\lambda] \geq \langle u_n-u_m, A(u_n-u_m)\rangle$$

But A is positive definite, which implies that $\{u_n\}$ is a Cauchy sequence. Since V is complete, therefore $u_n \to h$ as $n \to \infty$.

Suppose that V is separable so that there exists at least one linearly independent sequence $\{p_n\}$, p_n in V, so that the finite-dimensional, and hence closed, linear subspaces B_n spanned by $\{p_0, p_1, \ldots, p_{n-1}\}$ formed an expanding sequence with $\bigcup_0^\infty B_n = V$. As a practical matter in applying Theorem 3.2 we must solve the minimization problem

$$E(u_n) = \min\{E(u), u \in B_n\}$$

for each n, and the solution will be expressed as

$$u_n = \sum_{j=0}^{n} \alpha_{nj} p_j$$

where the coefficients α_{nj} depend on n. It would be convenient for this procedure if the coefficients would be independent of n, and this leads to the following topic.

(b) Conjugate Direction Method

Definition 3.1. Let p_m, p_n be non-zero elements in V. If

$$\langle p_m, Ap_n \rangle = 0, \ m \neq n$$

then p_m and p_n are said to be A-conjugate or A-orthogonal.

The iteration method in which the sequence of non-zero elements $\{p_n\}$ that is chosen to satisfy the A-conjugate condition is a conjugate direction method. The elements $\{p_0, p_1, \ldots\}$ may be determined before the iterative process, or the element p_n may be determined at the nth iteration. Let $\left\{\sum_{j=0}^{n} \alpha_{nj} p_j\right\}$ be an approximating sequence to the solution h. It follows from the definition above that the space spanned by p_n is A-orthogonal to the subspace B_n spanned by

$\{p_o, p_1, \ldots, p_{n-1}\}$, and consequently the coefficients α_{nj} are independent of n.

Lemma 3.1. Let $\{p_n\}$ be a sequence of elements in V whose elements are mutually A-orthogonal. Let B_n be the subspace spanned by $\{p_o, p_1, \ldots, p_{n-1}\}$. Suppose that

$$u_o = 0$$

$$u_{m+1} = u_m + \alpha_m p_m$$

$$\alpha_m = \frac{\langle -\nabla E(u_m), p_m \rangle}{2\langle p_m, Ap_m \rangle}$$

then u_n minimizes $E(u)$ on B_n.

Proof: It follows from $u_m = u_{m-1} + \alpha_{m-1} p_{m-1}$, $\langle \nabla E(u_m), p_j \rangle = \langle \nabla E(u_{m-1}), p_j \rangle + 2\alpha_{m-1} \langle Ap_{m-1}, p_j \rangle$. For $j < m-1$, we have $\langle \nabla E(u_m), p_j \rangle = \langle \nabla E(u_{m-1}, p_j \rangle$. By the definition of α_m, $\langle \nabla E(u_{j+1}), p_j \rangle = 0$. Hence $\langle \nabla E(u_m), p_j \rangle = 0$ for $j=1, \ldots, m-1$ or $\nabla E(u_m)$ is orthogonal to B_m, and the assertion follows from Theorem 3.1.

It is interesting to observe that if V is finite dimensional, say n, then $u_n = h$, and the iteration always converges in finitely many steps. Whenever $\langle \nabla E(u_m), p_m \rangle = 0$, $u_{m+1} = u_m$ and the minimum of $E(u)$ in B_m is also the minimum in B_{m+1}. This occurs, for example, when $u_m = h$. The assumption that the iteration starts with $u_o = 0$ is not essential. For if $u_o \neq 0$, consider the problem $A(u+u_o) = k$; the iteration $\hat{u}_o = 0$, $\hat{u}_{m+1} = \hat{u}_m + [\langle -\nabla E(u_m), p_m \rangle / 2\langle p_m, Ap_m \rangle] p_m$ then converges to the solution $\hat{h} = h - u_o$. As an immediate consequence of the above lemma and Theorem 3.3, we have the following.

Theorem 3.4. Let $\{p_n\}$ be a sequence of elements in V whose elements are mutually A-orthogonal. Let B_n be the subspace spanned by $\{p_0, p_1, \ldots, p_{n-1}\}$ and $V = \bigcup_{0}^{\infty} B_n$. Let u_0 be arbitrary, and

$$u_{m+1} = u_m + \alpha_m p_m$$

$$\alpha_m = \frac{\langle -\nabla E(u_m), p_m \rangle}{2\langle p_m, Ap_m \rangle} .$$

Then u_n converges to the solution h.

(c) The Method of Conjugate Gradients

In the discussion above on the conjugate direction method, the determination of the sequence of vectors $\{p_n\}$ was governed only by the requirement that they be A-conjugate; their determination remains relatively arbitrary. From the computational standpoint, it is convenient and frequently desirable to generate the p_n's at each step in the iteration process. The method now introduced is the algorithm used by Hestenes that generates a particularly useful set of conjugate directions. Each direction p_n is generated by "A-conjugate-izing" the gradient vector, and thus the name conjugate gradient is given.

The iteration is defined as follows:

let u_0 be arbitrary

$$\nabla E(u_0) = 2(Au_0 - k)$$

$$p_0 = -\nabla E(u_0) .$$

Having obtained u_n, $\nabla E(u_n)$, and p_n, the iteration is continued according to the expressions below.

$$\alpha_n = \frac{\langle -\nabla E(u_n), p_n \rangle}{2\langle p_n, Ap_n \rangle}$$

$$u_{n+1} = u_n + \alpha_n p_n \tag{3.3}$$

$$\nabla E(u_n) = 2(Au_{n+1} - k)$$

$$\beta_n = \frac{\langle \nabla E(u_n), Ap_n \rangle}{\langle p_n, Ap_n \rangle}$$

$$p_{n+1} = -\nabla E(u_{n+1}) + \beta_n p_n$$

Theorem 3.5. The quantities defined by the iteration process above satisfy the following relations:

$$\langle p_m, Ap_n \rangle = 0 \ , \ m \neq n$$

$$\langle p_m, \nabla E(u_n) \rangle = 0 \ , \ m < n \tag{3.4}$$

$$\langle p_m, \nabla E(u_n) \rangle = ||\nabla E(u_m)||^2 \ , \ m \geq n$$

$$\langle \nabla E(u_m), \nabla E(u_n) \rangle = 0 \ , \ m \neq n$$

Proof: From the defining expressions of α_n and β_n, the equalities above may be shown by induction.

Corollary: The conjugate gradient method is a special case of the conjugate direction method.

Proof: The assertion follows from expression (3.4) above.

If the set of vectors $\{p_o, p_1, \ldots\}$ generated above spans V, then the solution h would be achieved by Theorem 3.3. However, even for the cases in which the set $\{p_o, p_1, \ldots\}$ is not complete in V, nonetheless, we can still claim that u_n converges to h. This is an important and desirable fact of this method. The next theorem is devoted to demonstrating that point.

Lemma 3.2. α_n as defined by Equation (3.3) is bounded above by $||p_n||^2/2\langle p_n, Ap_n\rangle$.

Proof: $||p_n||^2 = ||\nabla E(u_n)||^2 + \beta_{n-1}^2 ||p_{n-1}||^2$

$\geq ||\nabla E(u_n)||^2$

Therefore,

$$\frac{||p_n||^2}{\langle p_n, Ap_n\rangle} = \frac{||\nabla E(u_n)||^2}{\langle p_n, Ap_n\rangle} \frac{||p_n||^2}{||\nabla E(u_n)||^2}$$

$$= 2\alpha_n \frac{||p_n||^2}{||\nabla E(u_n)||^2}$$

$$\geq 2\alpha_n$$

Lemma 3.3. Let $\sigma = \sup\left\{\lambda, \lambda \in \text{spectrum of the positive definite, self-adjoint, continuous linear operator } A\right\}$. Then

$$\frac{2}{\sigma} \leq \alpha_n$$

Proof: The lemma follows from the fact that

$$\sigma = \sup_{u \in V} \frac{\langle u, Au\rangle}{\langle u, u\rangle} \geq \frac{\langle p_n, Ap_n\rangle}{\langle p_n, p_n\rangle}$$

Lemma 3.4. E is a strictly decreasing function of n, i.e., $E(u_n) > E(u_{n+1})$, unless the solution is attained at the nth iteration.

Proof:

$$E(u_n) - E(u_{n+1}) = E(u_n) - E(u_n + \alpha_n p_n)$$

$$= -2\alpha_n\langle p_n, A(h-u_n)\rangle - \alpha_n^2\langle p_n, Ap_n\rangle$$

$$= \alpha_n \langle p_n, \nabla E(u_n)\rangle - \alpha_n^2 \langle p_n, Ap_n\rangle$$

$$= \alpha_n \langle p_n, \nabla E(u_n)\rangle - \alpha_n \frac{\langle p_n, \nabla E(u_n)\rangle}{\langle p_n, Ap_n\rangle} \langle p_n, Ap_n\rangle$$

$$= \frac{\alpha_n}{2} \langle p_n, \nabla E(u_n)\rangle$$

$$= \frac{\alpha_n}{2} ||\nabla E(u_n)||^2$$

The quantity $\frac{\alpha_n}{2} ||\nabla E(u_n)||^2$ is positive unless $u_n = h$, which is the assertion.

<u>Theorem 3.6</u>. The sequence $\{u_n\}$ obtained from the above conjugate gradient method converges to h, the solution to Equation (3.1).

<u>Proof</u>: In Theorem 3.3, we have shown that the sequence $\{u_n\}$ converges to some element in V, say u*. $\{E(u_n)\}$ is a monotonically decreasing sequence that is bounded below by zero, hence it converges. It follows from the expression in Lemma 3.4

$$\lim_{n \to \infty} \frac{\alpha_n}{2} ||\nabla E(u_n)||^2 = \lim_{n \to \infty} [E(u_n) - E(u_{n+1})]$$

$$= 0 .$$

But according to Lemma 3.3, $\alpha_n \geq \frac{2}{\sigma}$. Thus, $||\nabla E(u_n)|| = ||2A(h-u_n)||$ converges to zero as $n \to \infty$. By continuity, $A(h-u^*) = 0$. A, being positive definite, implies $u^* = h$.

3.3 <u>Nonlinear Theory</u>

Aside from the case of linear or quadratic functionals, when the Frechet differential of the functional is set equal to zero, the resulting equation is nonlinear. In this chapter we will consider the extension of the technique discussed previously for the linear

theory to solve the equation of the form

$$P(u) = 0$$

where P is a nonlinear operator mapping the real Hilbert space into itself. Suppose that u^* is the solution to the above operator equation. Then

$$\begin{aligned} 0 &= P(u^*) \\ &= P(u+(u^*-u)) \\ &= P(u)+F(u)(u^*-u)+o(||u^*-u||) \end{aligned}$$

where F(u) is the Frechet derivative. Therefore, for u sufficiently near the solution u^*, i.e., $||u^*-u||$ sufficiently small, we almost have a linear operator equation with the linear operator F(u). Since F(u) depends on u in general, if we generate conjugate directions with respect to F(u), we can at most assert that any two consecutive p_n vectors are F(u)-conjugate, while the other vectors are approximately F(u)-conjugate depending on how near u is to u^*.

Assume that the error functional E (or performance functional, as it is often called in control theory) defined on V possesses the following representation about u

$$E(u+h) = E(u)+\langle P(u),h\rangle + \frac{1}{2}\langle F(u)h,h\rangle + O(||h||^3) \qquad (3.5)$$

where P(u) and F(u) are the first and second Frechet derivatives, respectively. Suppose that E attains its minimum at u; then by Theorem 2.2, we can assert that the linear operator F(u) is positive definite in some neighborhood D about u. We will make the assumption that F(u) is self-adjoint. As we shall see in the following sections concerning the application of the method to control problems, the operator F(u) is indeed linear, positive definite, continuous and self-adjoint.

In view of the results above, we make the extension of the algorithm given previously as follows:

let u_o be arbitrary

$$G(u_o) = \nabla E(u_o)$$

$$p_o = -G(u_o)$$

Having obtained u_n, p_n and $\nabla E(u_n)$, the iteration is continued according to the expressions below:

$$u_{n+1} = u_n + \alpha_n p_n \tag{3.6}$$

where α_n is the smallest positive solution α of $\langle G(u_n+\alpha p_n), p_n\rangle = 0$,

$$G(u_{n+1}) = \nabla E(u_{n+1}) \tag{3.7}$$

$$\beta_n = \frac{\langle G(u_{n+1}), F(u_{n+1})p_n\rangle}{\langle p_n, F(u_{n+1})p_n\rangle} \tag{3.8}$$

$$p_{n+1} = -G(u_{n+1}) + \beta_n p_n \tag{3.9}$$

To determine the value of α that satisfies the equation $\langle G(u_n+\alpha p_n), p_n\rangle = 0$ is a difficult task in general, but we will make the following observations.

Lemma 3.5. Let D be a convex region in V containing u^* such that F(u) is positive definite. Then E is a convex functional on D.

Proof: Suppose $u_1, u_2 \in D$, then by convexity of D,

$tu_1+(1-t)u_2 \in D$ for $0 \leq t \leq 1$. The rest follows immediately from Equation (3.5) and the fact that the weighted sum of the squares is greater than or equal to the square of the weighted sums.

Lemma 3.6. If E is a convex functional on a subset W of V, then $\{u: E(u) \leq M, M \text{ being a positive constant}\} \subset W$ is convex.

Proof: $$E(tu_1+(1-t)u_2) \leq tE(u_1)+(1+t)E(u_2)$$
$$\leq M$$

which implies the assertion.

As the consequence of the above lemmas, we have the following result.

Theorem 3.7. If F(u) is positive definite, then the value of α that minimizes $E(u_n+\alpha p_n)$ coincides with the value of α that satisfies $\langle G(u_n+\alpha p_n), p_n\rangle = 0$.

We have thus reduced the problem of finding the solution to $\langle G(u_n+\alpha p_n), p_n\rangle = 0$ to a one-dimensional minimization problem.

Theorem 3.8. Suppose that F(u) is uniformly bounded and uniformly positive definite in Q, the closure of $\{u: E(u) \leq E(u_o)\}$. For the sequence $\{u_n\}$ generated by Equations (3.6) to (3.9), $\{G(u_n)\}$ converges to zero.

Proof: Since F(u) is uniformly positive definite in Q, then there exists positive constants m and M so that $\theta < mI \leq F(u) \leq MI$, where I is the identity operator, and θ is the null operator. Let α_n be the value of α that minimizes $E(u_n+\alpha p_n)$; then $\langle G(u_n+\alpha p_n), p_n\rangle = 0$ and consequently

$$\langle G(u_n+\alpha_n p_n), p_{n+1}\rangle = \langle G(u_n+\alpha_n p_n), G(u_n+\alpha_n p_n)+p_n\rangle$$

$$= ||G(u_n+\alpha_n p_n)||^2$$

$$\frac{\partial E}{\partial \alpha}(u_n+\alpha p_n)\Big|_{\alpha=0} = -\langle G(u_n+\alpha_n p_n), p_n\rangle$$

$$= ||G(u_n)||^2$$

$$\frac{\partial^2 F}{\partial \alpha^2}(u_n+\alpha p_n) = \langle F(u_n+\alpha p_n)p_n, p_n\rangle$$

$$< M||p_n||^2$$

whenever $u_n+\alpha p_n$ is in Q. Since

$$\frac{\partial E}{\partial \alpha}(u_n+\alpha p_n)\Big|_{\alpha=\alpha_n} = 0$$

therefore

$$\alpha_n M||p_n||^2 > ||G(u_n)||^2$$

From Equation (3.9) and the fact that the consecutive elements in the sequence $\left\{p_n\right\}$ are F(u)-conjugate thus

$$m||p_n||^2 \leq M||G(u_n)||^2$$

and $\alpha_n \geq m/M^2$. From

$$E(u_n+\alpha p_n) = E(u_n)+\alpha\langle G(u_n), p_n\rangle + \frac{\alpha^2}{2}\langle G(u_n+t\alpha p_n)p_n, p_n\rangle$$

for some $t \in [0,1]$, for $\alpha \leq \alpha_n$

$$E(u_n+\alpha p_n) \leq E(u_n)-\alpha||G(u_n)||^2 + \frac{1}{2}\alpha^2 M||p_n||^2$$

$$\leq E(u_n)-\alpha||G(u_n)||^2 + \frac{1}{2}\alpha^2\frac{M^2}{m}||G(u_n)||^2$$

In particular, consider $\alpha = m/M$, then

$$E(u_{n+1}) \leq E(u_n + \frac{m}{M} p_n)$$

$$\leq E(u_n) - \frac{m}{M^2} ||G(u_n)||^2 + \frac{1}{2}\frac{m}{M^2} ||G(u_n)||^2$$

$$\leq E(u_n) - \frac{1}{2}\frac{m}{M^2} ||G(u_n)||^2$$

Therefore, $\{E(u_n)\}$ is a monotonically decreasing sequence, and hence convergent with the assumption that $E(u)$ is bounded below. $\{E(u_{n+1})-E(u_n)\} \to 0$ as $n \to \infty$ implies that $||G(u_n)|| \to 0$ as $n \to \infty$.

As a consequence of the uniformly positive definite property of $F(u)$ in Q, and supposing that Q is compact, we then have the following result.

Theorem 3.9. The sequence $\{u_n\}$ converges to a unique element u^* in Q, the solution to the minimization problem.

3.4 Remarks

(a) In finding the value of α that minimizes $E(u_n+\alpha p_n)$, we may use the expression α given in the linear case, namely,

$$\alpha = -\frac{\langle G(u_n), p_n\rangle}{2\langle p_n, F(u_n)p_n\rangle} \tag{3.10}$$

as a first order approximation to guide the initial search. The quantities p_n, $G(u_n)$ and $F(u_n)$ are already available in the computational process, thus the evaluation of α does not involve much work. For most problems, it is expected that this approximation would get better as u_n gets closer to u^*, for then the vectors would be closer to mutually $F(u^*)$-conjugate.

(b) Theorem 3.8 indicates that it is desirable to select the initial estimate of the solution u_o such that the second Frechet derivative of E at u_o is uniformly bounded and uniformly positive definite. We may use this fact as a guide to select the initial approximation.

(c) In optimal control with state variable constraint applications, the set of admissible controls (this term will be made precise later) in general will not form a linear subspace nor even meet the convexity hypothesis in the discussion above. For the problem in Section VI, the admissible control set possesses the necessary conditions being convex, while for the problem in Section VII it does not. The modifications made to obtain convergence in computation to the desired solution are presented in detail there. The convergence is along the expanding sequence of sets $\left\{B_n \cap Q\right\}$ where Q denotes the set of admissible controls.

(d) If β_n is set equal to zero in each step, the direction of search p_n would be along the negative direction of the gradient of E at u_n, and if α_n is selected so that the performance functional is minimized, then this is the well-known method of steepest descent. It is worthy to note that in the method of steepest descent, the performance functional is not minimized in a sequence of expanding subspaces as it is in the conjugate direction methods.

(e) At any step of the iteration process, we can start anew with only a small amount of labor involved, keeping the approximation last obtained as the initial estimate.

(f) Other variations of the conjugate gradient algorithm when

F(u) is independent of u may be found in the papers of Hestenes and Stiefel [7], [8] where the development is presented in great detail.

SECTION IV

THE CLASS OF CONTROL PROBLEMS TO BE CONSIDERED

Our ultimate goal is to apply the technique develope in the previous section to solve the class of control problems which we formulate below. Supoose that the dynamical system is governed by the differential equation

$$\frac{d\underline{x}}{dt} = f(\underline{x},x) \tag{4,1}$$

where $\underline{x}$ is a real n-vector for each t, called the state of the system; u is a real m-vector for each t, called the control vector; and f is a real n-vector for each t, that is twice continuously differentiable in its arguments (t will be interpreted as time with values in $E^{1\delta}$). Let $\underline{x}(t_0)$ be the initial state of the system, and let it be desired to transfer the system from the given initial state to some final state lying on a smooth hypersurface

$$\psi\left[\underline{x}(t_f)\right] = 0 \tag{4,2}$$

where the terminal time t_f is not fixed, while the states are confined to whithin a closed region in E^n given by the inequalities

$$g_k(\underline{x}) \leq 0, \ k=1,2,\ldots,N \tag{4.3}$$

where g_k is an m-time continuous differentiable function of $\underline{x}$. We will callla control u an element in the Hilbert space of piecewise continuous functions on $\left[t_0, t_f\right]$ with inner product defined as

---

$^{\delta}E^m$ denotes an m-dimensional Euclidean space

$$\langle u_1, u_2 \rangle = \int_{t_o}^{t_f} u_1(t) u_2(t) dt$$

admissible if the corresponding trajectory in E^n does not violate the state constraints above for all $t \in [t_o, t_f]$. Denote the class of admissible controls by Q. Let the performance functional be

$$E(u) = w[\underline{x}(t_f)] + \int_{t_o}^{t_f} L(u, \underline{x}) dt \tag{4.4}$$

or alternatively as in the formulation of Mayer,

$$E(u) = \phi[x(t_f)]$$

a function of end values of the states, where $x(t_f)$ is an augmented (n+1)-vector. In the following x will be used to denote either the n-vector or the augmented (n+1)-vector without further specifying whenever the situation is clear from the context.

The problem's objective is to find the control u in Q that minimizes the performance functional while satisfying the conditions (4.1), (4.2) and (4.3). We will make an assumption that there exists a unique solution to this minimization problem.

SECTION V

COMPUTATIONAL CONSIDERATIONS

5.1 Nomenclature

For each control u in Q, there corresponds a trajectory in E^n. It may consist of two types of arcs. The portion of a trajectory in which the states satisfy

$$g_k(x) < 0 \qquad k=1,2,\ldots,N \tag{5.1}$$

will be called an interior arc, and the portion that satisfies

$$g_k(x) = 0 \tag{5.2}$$

for some k, k=1,2,...,N will be called a boundary arc. A trajectory may comprise entirely a boundary arc, an entirely interior arc, or a combination of interior arcs and boundary arcs as shown in Figures 5.1 to 5.3.

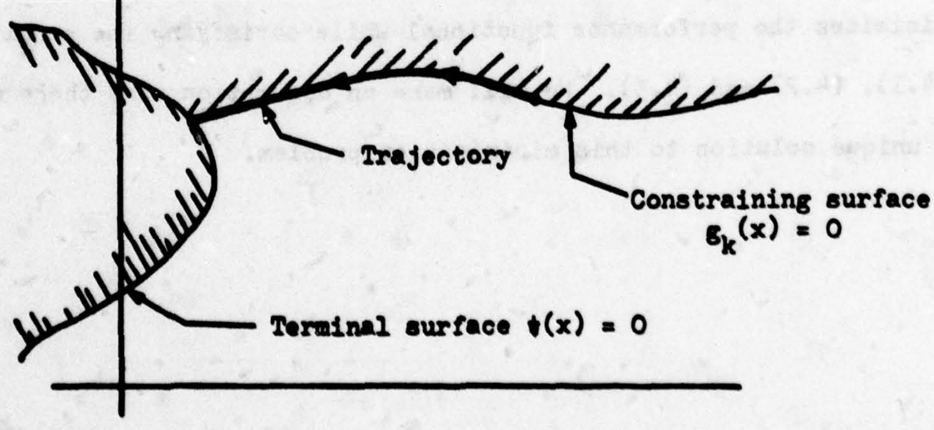


Figure 5.1. Trajectory comprises only boundary arc

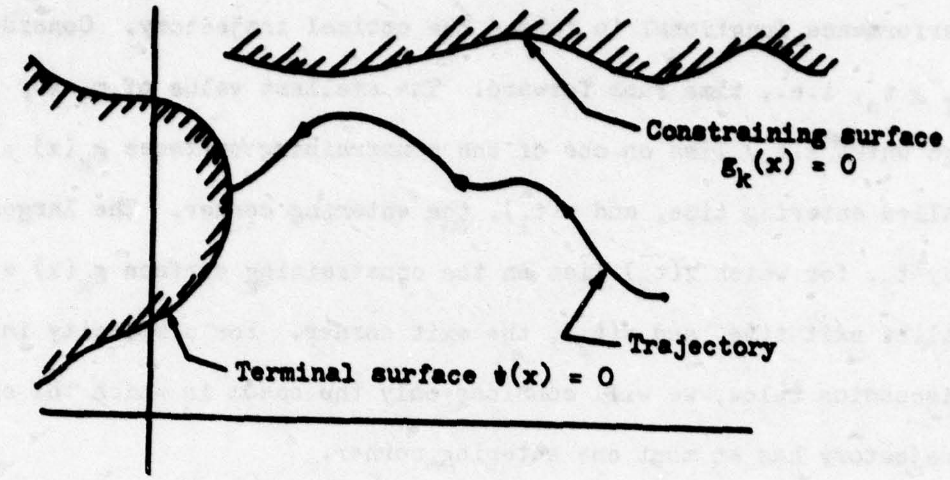


Figure 5.2. Trajectory comprises only interior arc

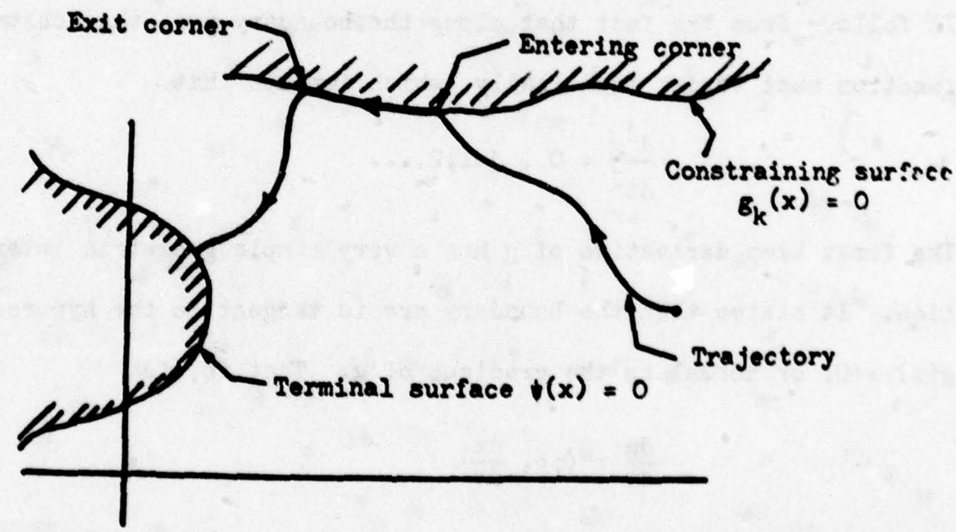


Figure 5.3. Trajectory comprises interior arcs and boundary arc

The trajectory corresponding to the control $u^* \in Q$ that minimizes the performance functional is termed the optimal trajectory. Consider $t_f \geq t_o$, i.e., time runs forward. The smallest value of t, say t_1, for which $x(t_1)$ lies on one of the constraining surfaces $g_k(x) = 0$ is called entering time, and $x(t_1)$, the entering corner. The largest t, say t_2, for which $x(t_2)$ lies on the constraining surface $g_k(x) = 0$ is called exit time, and $x(t_2)$, the exit corner. For simplicity in the discussion below, we will consider only the cases in which the optimal trajectory has at most one entering corner.

5.2 Control on the Boundary Arc

For the period $t \in [t_1, t_2]$ along the boundary arc, the states are interrelated by

$$g_k(x) = 0^{5} \tag{5.3}$$

It follows from the fact that along the boundary arc, the constraint function must vanish identically, which implies that

$$\frac{d^j g}{dt^j} = 0 \ , \ j=1,2,\ldots \tag{5.4}$$

The first time derivative of g has a very simple geometric interpretation. It states that the boundary arc is tangent to the hypersurface $g(x) = 0$, or normal to the gradient of g. That is,

$$\frac{dg}{dt} = \langle \nabla g, \frac{dx}{dt} \rangle \tag{5.5}$$

The control u will be determined according to (5.5) if u appears

[5] If two or more constraints are involved for $t \in [t_1, t_2]$, the argument is similar, and the subscript k on g will be dropped in subsequent discussion.

explicitly in the expression. If it does not, we may consider the second derivative or higher derivatives so that u will appear explicitly in $\frac{d^N g}{dt^N} = 0$. If the system is controllable [15], the existence of a smallest integer N, the order of the derivative of g for which u appears explicitly is assured. From (5.4) in particular for $t = t_1$, we have

$$g[x(t_1)] = 0 \tag{5.6}$$

$$\frac{d^j g}{dt^j}[x(t_1)] = 0 \qquad j=1,\ldots,N-1 \tag{5.7}$$

It is worthy to note that Equations (5.6) and (5.7) along with the control u satisfying

$$\frac{d^N g}{dt^N} = 0 \tag{5.8}$$

imply that

$$\frac{d^j g}{dt^j} = 0 \qquad j=0,1,\ldots,N \tag{5.9}$$

for all $t \in [t_1, t_2]$.

We will make the necessary assumptions such as g has no singular point, i.e., $\nabla g(x) \neq 0$, to permit a possible unique solution for u in terms of the states in (5.8). Actually, we need only to make such assumptions along the optimal boundary arc. But from the computational point of view, in particular using the conjugate gradient technique, since we have no advance knowledge of the whereabouts of the boundary arc on the constraint surface, the above provisions are necessary.

5.3 The Perturbation Equations

Suppose that $u^* \in Q$ is the optimal control of our minimization problem, and that x^* is the corresponding optimal trajectory. Consider

$u = u^* + \theta h \in Q$, where θ is a real number and h a piecewise continuous function. Let

$$x(t) = x^*(t) + \theta z(t) \tag{5.10}$$

be the trajectory generated by u. It follows from (5.10) that we have

$$\frac{dx(t)}{dt} = \frac{dx^*(t)}{dt} + \theta \frac{dz}{dt} \, . \tag{5.11}$$

Since x is a solution to (4.1),

$$\begin{aligned} \frac{dx(t)}{dt} &= f[x(t),u(t)] \\ &= f[x^*(t)+\theta z(t),u^*(t)+\theta h(t)] \\ &= f[x^*(t),u^*(t)] + \frac{\partial f}{\partial x}\Big|_* \theta z(t) + \frac{\partial f}{\partial u}\Big|_* \theta h(t) + o(\theta) \end{aligned} \tag{5.12}$$

where $\frac{\partial f}{\partial x}$ is the Jacobian matrix of f with respect to x, $\frac{\partial f}{\partial u}$ is a row vector. The symbol $|_*$ indicates that the quantity is to be evaluated along the optimal trajectory. It follows then from (5.10) and (5.12), and ignoring the factor $o(\theta)$ (this will not alter the ultimate outcome when the limit of θ approaches zero is taken), that z is a solution to the linear differential equation

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\Big|_* z + \frac{\partial f}{\partial u}\Big|_* h \tag{5.13}$$

and

$$z(t_o) = 0 \, .$$

On the boundary arc, the control and states are further subject to $\frac{d^N g}{dt^N} = 0$. Denote $\frac{\partial^N g}{\partial t^N}$ by G.

$$G[x(t),u(t)] = G[x^*(t)+\theta z(t),u^*(t)+\theta h(t)]$$

$$= G(x^*,u^*)+\theta \left\langle \frac{\partial G}{\partial x}\Big|_*, z\right\rangle + \frac{\partial G}{\partial u}\Big|_* \theta h+o(\theta) \tag{5.14}$$

Again, ignoring the factor $o(\theta)$ we have

$$\left\langle \frac{\partial G}{\partial x}\Big|_*, z\right\rangle + \frac{\partial G}{\partial u}\Big|_* h = 0 \tag{5.15}$$

If $\frac{\partial G}{\partial u}\Big|_* \neq 0$, then it follows from (5.13) and (5.15) that

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\Big|_* z - \frac{\partial f}{\partial u}\Big|_* \left[\frac{\partial G}{\partial u}\right]^{-1} \left\langle \frac{\partial G}{\partial x}\Big|_*, z\right\rangle \tag{5.16}$$

on the boundary arc.

5.4 The First Frechet Differential of the Performance Functional

Assume that the performance functional E as defined below satisfies the conditions in Theorem 2.1, then we may evaluate its Frechet derivative by formula (2.3), and ultimately obtain the gradient of E. Let λ be a piecewise differentiable n-vector-valued function of t as yet unspecified. We will call $\lambda(t)$ a costate vector. Let

$$H(x,\lambda,u) = \langle \lambda, f(x,u)\rangle \tag{5.17}$$

and call the scalar function H the Hamiltonian of the system. Treating the conditions (4.1), (4.2), (5.6), and (5.7) as constraints, consider the performance functional E at $u = u^*+\theta h \in Q$.

$$E(u) = \phi[x(t_f)]+\psi[x(t_f)]+\langle \mu, S[x(t_1)]\rangle + \int_{t_o}^{t_f} [H(x,\lambda,u)-\langle \lambda, \frac{dx}{dt}\rangle]dt \tag{5.18}$$

where μ is a constant N-vector and

$$S[x(t_1)] = \begin{bmatrix} g[x(t_1)] \\ \frac{dg}{dt}[x(t_1)] \\ \vdots \\ \frac{d^{N-1}g}{dt^{N-1}}[x(t_1)] \end{bmatrix} \tag{5.19}$$

for convenience. Making expansions about the optimal trajectory,

$$\phi[x(t_f)] = \phi[x^*(t_f^*)] + \langle \frac{\partial \phi}{\partial x}\Big|_*, dx(t)\rangle_{t_f} + o(\theta) \tag{5.20}$$

$$\psi[x(t_f)] = \psi[x^*(t_f^*)] + \langle \frac{\partial \psi}{\partial x}\Big|_*, dx(t)\rangle_{t_f} + o(\theta) \tag{5.21}$$

$$S[x(t_1)] = S[x^*(t_1^*)] + \frac{\partial S}{\partial x}\Big|_{*t_1} dx(t_1) + o(\theta) \tag{5.22}$$

For the functional E at u^*, we have

$$E(u^*) = \phi[x^*(t_f^*)] + \psi[x^*(t_f^*)] + \langle \mu, S[x^*(t_1^*)\rangle + \int_{t_o}^{t_f^*} [H(x^*,\lambda^*,u^*) - \langle \lambda^*, \frac{dx^*}{dt}\rangle]dt \tag{5.23}$$

Therefore,

$$E(u)-E(u^*) = \langle \frac{\partial \phi}{\partial x}\Big|_*, dx(t)\rangle_{t_f} + \langle \frac{\partial \psi}{\partial x}\Big|_*, dx(t)\rangle_{t_f} + \langle \mu, \frac{\partial S}{\partial x}\Big|_{*t_1}\rangle dx(t_1)$$

$$+ \int_{t_o}^{t_f} [H(x,\lambda,u) - H(x^*,\lambda^*,u^*) - \theta\langle \lambda, \frac{dz}{dt}\rangle]dt + o(\theta) \tag{5.24}$$

To take into account the possible discontinuities at the entering and exit corners, the above integral will be written as three integrals over the intervals $[t_o, t_1)$, (t_1, t_2) and $(t_2, t_f]$. After integration by parts is performed, (5.24) becomes

$$E(u)-E(u^*) = \langle \frac{\partial \phi}{\partial x}\Big|_*, dx(t)\rangle_{t_f} + \langle \frac{\partial \psi}{\partial x}\Big|_*, dx(t)\rangle_{t_f} + \langle \mu, \frac{\partial S}{\partial x}\Big|_{*t_1} dx(t_1)\rangle$$

$$+\theta\langle\lambda(t_o),z(t_o)\rangle-\theta\langle\lambda(t_1^-),z(t_1^-)\rangle+\theta\int_{t_o}^{t_1^-}[\langle\frac{d\lambda}{dt},z\rangle+\langle\frac{\partial H}{\partial x}\Big|_*,z\rangle+\frac{\partial H}{\partial u}\Big|_* h]dt$$

$$+\theta\langle\lambda(t_1^+),z(t_1^+)\rangle-\theta\langle\lambda(t_2^-),z(t_2^-)\rangle+\theta\int_{t_1^+}^{t_2^-}\Big\{\langle\frac{\partial\lambda}{\partial t},z\rangle+\langle\frac{\partial H}{\partial x}\Big|_*,z\rangle$$

$$+\frac{\partial H}{\partial u}\Big|_*[\frac{\partial G}{\partial u}]^{-1}\Big|_*\langle\frac{\partial G}{\partial x}\Big|_*,z\rangle\Big\}dt$$

$$+\theta\langle\lambda(t_2^+),z(t_2^+)\rangle-\theta\langle\lambda(t_f),z(t_f)\rangle+\theta\int_{t_2^+}^{t_f}[\langle\frac{\partial\lambda}{\partial t},z\rangle+\langle\frac{\partial H}{\partial x}\Big|_*,z\rangle+\frac{\partial H}{\partial u}\Big|_* h]dt$$

$$+ o(\theta) \tag{5.25}$$

But $\quad dx(t_j) = \theta z(t_j) + \frac{dx}{dt}(t_j)\tau_j + o(\theta) \qquad (5.26)$

where $\quad \tau_j = t_j - t_j^*$, $j=1,2,f$

and $\quad z(t_o) = 0$

Thus

$$E(u)-E(u^*) = \langle \frac{\partial \phi}{\partial x}\Big|_{*t_f} + \frac{\partial \psi}{\partial x}\Big|_{*t_f} - \lambda(t_f), dx(t_f)\rangle + \langle\lambda(t_f), \frac{dx(t_f)}{dt}\tau_f\rangle$$

$$+\langle\lambda(t_2^+)-\lambda(t_2^-),dx(t_2)\rangle-\langle\lambda(t_2^+),\frac{dx(t_2^+)}{dt}\tau_2\rangle$$

$$+\langle\lambda(t_2^-),\frac{dx(t_2^-)}{dt}\tau_2\rangle-\langle\lambda(t_1^+),\frac{dx(t_1^+)}{dt}\tau_1\rangle+\langle\frac{\partial S}{\partial x}\Big|_{*t_1}\mu-\lambda(t_1^-)+\lambda(t_1^+)\rangle$$

$$+\theta\int_{t_o}^{t_1^-}[\langle\frac{d\lambda}{dt},z\rangle+\langle\frac{\partial H}{\partial x}\Big|_*,z\rangle+\frac{\partial H}{\partial u}\Big|_* h]dt$$

$$+\theta\int_{t_1^+}^{t_2^-}\left\{\langle\frac{\partial\lambda}{\partial t},z\rangle+\langle\frac{\partial H}{\partial x}\Big|_*,z\rangle+\frac{\partial H}{\partial u}\Big|_*\left[\frac{\partial G}{\partial u}\right]^{-1}\Big|_*\langle\frac{\partial G}{\partial x}\Big|_*,z\rangle\right\}dt$$

$$+\theta\int_{t_2^+}^{t_f}\left[\langle\frac{\partial\lambda}{\partial t},z\rangle+\langle\frac{\partial H}{\partial x}\Big|_*,z\rangle+\frac{\partial H}{\partial x}\Big|_* h\right]dt+o(\theta) \qquad (5.27)$$

We are now going to impose the following conditions on the costate λ.

(a) Demand that λ be a solution of the differential equation

$$\frac{d\lambda}{dt}+\frac{\partial H}{\partial x}=0^{\S} \qquad (5.28)$$

for $t \in [t_o,t_1)$ and $t \in (t_2,t_f]$.

(b) On the boundary arc, for $t \in (t_1,t_2)$, require λ to satisfy

$$\frac{d\lambda}{dt}+\frac{\partial f}{\partial x}\lambda+\langle\frac{\partial f}{\partial u},\lambda\rangle\left[\frac{\partial G}{\partial u}\right]^{-1}\frac{\partial G}{\partial x}=0 \qquad (5.29)$$

(c) At t_1, t_2 and t_f demand that

$$\lambda(t_1^-)=\lambda(t_1^+)+\frac{\partial S}{\partial x}\Big|_{t_1}\mu \qquad (5.30)$$

$$\langle\lambda,\frac{dx}{dt}\rangle_{t_1^-}=\langle\lambda,\frac{dx}{dt}\rangle_{t_1^+} \qquad (5.31)$$

$$\lambda(t_2^-)=\lambda(t_2^+) \qquad (5.32)$$

$$\langle\lambda,\frac{dx}{dt}\rangle_{t_2^-}=\langle\lambda,\frac{dx}{dt}\rangle_{t_2^+} \qquad (5.33)$$

$$\lambda(t_f)=\frac{\partial\phi}{\partial x}(t_f)+\frac{\partial\psi}{\partial x}(t_f) \qquad (5.34)$$

$\S$ The symbol $|_*$ after the quantities $\frac{\partial H}{\partial x}$, etc. will be deleted henceforth.

Equations (5.31) and (5.33) indicate that the Hamiltonian is continuous at the entering and exit corners, and Equations (5.32) and (5.30) show that the costate is continuous at the exit corner while it is discontinuous at the entering corner with a jump equal to $\left.\frac{\partial S}{\partial x}\right|_{t_1} \mu$.

With the above conditions, the Frechet differential is

$$G(u^*)h = \int_{t_o}^{t_1^-} \frac{\partial H}{\partial u} hdt + \int_{t_2^+}^{t_f} \frac{\partial H}{\partial u} hdt \tag{5.35}$$

and consequently, the gradient of the performance functional

$$\nabla E(u) = \frac{\partial H}{\partial u}(x,\lambda,u) \tag{5.36}$$

for $t \varepsilon [t_o, t_1^-)$ and $t \varepsilon (t_2^+, t_f]$.

5.5 The Second Frechet Differential of the Performance Functional

Suppose that for the control $u = u^* + \theta h \varepsilon Q$, the corresponding trajectory is

$$x(t) = x^*(t) + \theta z(t) + \theta^2 \omega(t) \tag{5.37}$$

instead of (5.10), where z again satisfies the differential equation (5.13) with $z(t_o) = 0$ and $\omega(t_o) = 0$. In view of (5.37), let us now re-examine the expression (5.24). The expression (5.20) becomes

$$\phi[x(t_f)] = \phi[x^*(t_f^*)] + \langle \frac{\partial \phi}{\partial x}(t_f), dx(t_f) \rangle + \frac{1}{2}\langle dx(t_f), \frac{\partial^2 \phi}{\partial x^2}(t_f) dx(t_f) \rangle + o(\theta^2) \tag{5.38}$$

where

$$dx(t_f) = \theta z(t_f) + \theta^2 \omega(t_f) + \theta \frac{dx(t_f)}{dt} \tau_f \tag{5.39}$$

Similarly, expressions (5.21) and (5.22) become

$$\phi[x(t_f)]=\phi[x^*(t_f^*)]+\langle\frac{\partial\phi}{\partial x}(t_f),dx(t_f)\rangle+\frac{1}{2}\langle dx(t_f),\frac{\partial^2\phi}{\partial x^2}(t_f),dx(t_f)\rangle+o(\theta^2) \tag{5.40}$$

and

$$S[x(t_1)]=S[x^*(t_1^*)]+\frac{\partial S}{\partial x}(t_1)dx(t_1)+\frac{1}{2}\langle dx(t_1),\frac{\partial^2 S}{\partial x^2}dx(t_1)\rangle+o(\theta^2) \tag{5.41}$$

The integrals in (5.25) become

$$-\int_{t_o}^{t_1^-}\langle\lambda,\frac{d}{dt}(\theta z+\theta^2\omega)\rangle dt = -\theta\langle\lambda(t_1^-),dx(t_1)+\frac{dx(t_1^-)}{dt}\tau_1\rangle$$

$$+\theta\int_{t_o}^{t_1^-}\langle\frac{d\lambda}{dt},z\rangle dt+\theta^2\int_{t_o}^{t_1^-}\langle\frac{d\lambda}{dt},\omega\rangle dt+o(\theta^2) \tag{5.42}$$

$$-\int_{t_2^+}^{t_f}\langle\lambda,\frac{d}{dt}(\theta z+\theta^2\omega)\rangle dt = \theta\langle\lambda(t_2^+),dx(t_2^+)+\frac{dx(t_2^+)}{dt}\tau_2\rangle$$

$$-\theta\langle\lambda(t_f),dx(t_f)+\frac{dx(t_f)}{dt}\tau_f\rangle+\theta\int_{t_2^+}^{t_f}\langle\frac{d\lambda}{dt},z\rangle dt+\theta^2\int_{t_2^+}^{t_f}\langle\frac{d\lambda}{dt},\omega\rangle dt$$

$$+o(\theta^2) \tag{5.43}$$

$$-\int_{t_1^+}^{t_2^-}\langle\lambda,\frac{d}{dt}(\theta z+\theta^2\omega)\rangle dt=\theta\langle\lambda(t_1^+),dx(t_1)\rangle+\frac{dx(t_1^+)}{dt}\tau_1\rangle$$

$$-\theta\langle\lambda(t_2^-),dx(t_2)+\frac{dx(t_2)}{dt}\tau_2\rangle+\theta\int_{t_1^+}^{t_2^-}\langle\frac{d\lambda}{dt},z\rangle dt+\theta^2\int_{t_1^+}^{t_2^-}\langle\frac{d\lambda}{dt},\omega\rangle dt+o(\theta^2) \tag{5.44}$$

and

$$H(x,\lambda,u)-H(x^*,\lambda^*,u^*)=\theta\langle\frac{\partial H}{\partial x},z\rangle+\theta^2\langle\frac{\partial H}{\partial x},\omega\rangle+\theta\langle\frac{\partial H}{\partial u},h\rangle$$

$$+\frac{\theta^2}{2}\left\langle\begin{bmatrix} z \\ h\end{bmatrix},\begin{bmatrix}\frac{\partial^2 H}{\partial x^2} & \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u}\right) & \frac{\partial^2 H}{\partial u^2}\end{bmatrix}\begin{bmatrix} z \\ h\end{bmatrix}\right\rangle + o(\theta^2) \qquad (5.45)$$

From Equations (5.38) to (5.45), in view of the conditions imposed on the costate λ and that $\frac{\partial H}{\partial u} = 0$ along the optimal trajectory, we have for the second Frechet differential

$$\langle h,F(u)h\rangle = \langle z(t_f),\frac{\partial^2\phi}{\partial x^2}z(t_f)\rangle + \langle z(t_f),\frac{\partial^2\psi}{\partial x^2}z(t_f)\rangle$$

$$+\langle z(t_1),\frac{\partial^2 S}{\partial x^2}z(t_1)\rangle+\int_{t_o}^{t_1}+\int_{t_2}^{t_f}\left\langle\begin{bmatrix} z \\ h\end{bmatrix}\begin{bmatrix}\frac{\partial^2 H}{\partial x^2} & \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u}\right) & \frac{\partial^2 H}{\partial u^2}\end{bmatrix}\begin{bmatrix} z \\ h\end{bmatrix}\right\rangle dt$$

(5.46)

5.6 An Approximation for the Hessian

The gradient of the performance functional follows immediately from the first Frechet differential. However, some further steps are necessary from (5.46) to obtain the Hessian of E which is to be used in the conjugate gradient algorithm. Recall that z is the solution to the linear time-varying differential equation

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\Big|_* z + \frac{\partial f}{\partial u}\Big|_* h \qquad (5.47)$$

for the unconstrained region given by

$$z(t) = L(t,t_j)z(t_j) + \int_{t_j}^{t} L(t,\tau)\frac{\partial f}{\partial u}(\tau)\Big|_* h(\tau)d\tau \tag{5.48}$$

where L is the fundamental matrix and $t_j = t_o, t_2^+$. $z(t_o) = 0$, but $z(t_2)$ as yet is not known. Writing (5.48) as

$$z = Th \tag{5.49}$$

and letting

$$W = \begin{bmatrix} T \\ I \end{bmatrix} \tag{5.50}$$

Formally, we have

$$\left\langle \begin{bmatrix} z \\ h \end{bmatrix}, \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u}\right) & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} \begin{bmatrix} z \\ h \end{bmatrix} \right\rangle = \left\langle Wh, \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u}\right) & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} Wh \right\rangle$$

$$= \left\langle h, W^{\neq} \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u}\right) & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} Wh \right\rangle^{\S} \tag{5.51}$$

$$\langle z(t_f), \frac{\partial^2 \phi}{\partial x^2}\Big|_* z(t_f)\rangle = \langle Th, \frac{\partial^2 \phi}{\partial x^2} Th\rangle_{t_f}$$

$$= \langle h, T^{\neq} \frac{\partial^2 \phi}{\partial x^2} Th\rangle_{t_f} \tag{5.52}$$

and similar expressions for the other factors in (5.46). The second

---

§ $\neq$ denotes adjoint.

Frechet differential now becomes

$$\langle h, F(u)h\rangle = \langle h, T^{\neq} \frac{\partial^2 \phi}{\partial x^2} Th\rangle_{t_f} + \langle h, T^{\neq} \frac{\partial^2 \psi}{\partial x^2} Th\rangle_{t_f} + \langle h, T^{\neq} \frac{\partial^2 S}{\partial x^2} Th\rangle_{t_1} + \int_{t_o}^{t_1}$$

$$+ \int_{t_2}^{t_f} \left\langle h, W^{\neq} \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u}\right) & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} Wh \right\rangle dt \qquad (5.53)$$

from which we can obtain the Hessian of the performance functional. Since (5.47) describes a time-varying linear system and the initial state is either at rest, $z(t_o) = 0$, or $z(t_2)$ which is small, (when u is sufficiently near u*), a "small" h will generate a "small" z (the converse is not necessarily true) and the term $\langle h, \frac{\partial^2 H}{\partial u^2} h\rangle$ is the "most" dependent on h in (5.53) [15]. Furthermore, $\frac{\partial^2 H}{\partial u^2}$ is positive definite [15], and therefore, by continuity there exists a region about u* for which $\frac{\partial^2 H}{\partial u^2}$ is positive definite.

In the discussion in Section III, Theorem 3.8, the only requirement on F(u) was that it must be uniformly positive definite in the neighborhood of u* to assure convergence of the iteration process (assuming also that the controls in this neighborhood are all admissible). Hence we may consider constructing a set of search directions $\{p_n\}$ conjugate with respect to $\frac{\partial^2 H}{\partial u^2}$, or locally conjugate to be precise since $\frac{\partial^2 H}{\partial u^2}$ depends on u in general. This simplification provides a considerable reduction in computation time and programming, and we will use this approximation below in our computations.

5.7 Computation for the Costate

In order to obtain the Hamiltonian at the nth iteration on which the gradient and Hessian are based, we must have the state variable and the costate in addition to the control u_n chosen for that iteration. Since the state variable is continuous and the initial condition is given, solving Equation (4.1) is a straightforward problem, provided the estimated quantities t_1, t_2 and t_f are settled. We will elaborate this point in the next subsection. On the other hand, the determination of the costate requires more considerations. Since the boundary condition for the costate is specified at the terminal time t_f and the costate is continuous at the exit corner, thus $\lambda(t)$ for $t \in (t_1^+, t_f]$ may be determined simply by solving the differential equations (5.28) and (5.29) backward in time using the latest estimated control and state variable. At the entering corner, when $t = t_1$, the costate may be discontinuous. In principle, it is possible to determine $\lambda(t_1^-)$, μ, t_1, a total of N+n+1 unknown quantities, from Equations (5.30), (5.31), (5.6) and (5.7) as long as these equations are independent. Since in any stage of the iteration process, the time at which the trajectory reaches the constraint surface t_1 is in general not equal to t_1^*, hence an exact solution to the above quantities is not really essential provided that some means are taken so that these quantities would converge to the desired values as the process progresses. Initially, a trial and error technique may be used to obtain an approximation to these quantities. Depending on the problem at hand, frequently some intuition as to the physical nature of the problem may serve as a guide to the guess and the method by which to improve the estimates at each step. This is the most difficult part of the computation and also one of the most time-consuming portions of the iteration process.

After the estimate of $\lambda(t_1^-)$ is selected, the costate in the remaining portion, for $t \in [t_o, t_1)$, may again resort to solving the differential equation (5.28).

5.8 Entering Time and Exit Time

Since the control program is updated at each step according to

$$u_{n+1} = u_n + \alpha_n p_n \tag{5.54}$$

the new trajectory may reach the constraint surface sooner or later than the previous iteration. In other words, the entering time in general varies with each iteration, and it is dictated by the control chosen. If $t_1^{(n)}$ ‡ is larger than $t_1^{(n+1)}$, there is no problem. However, when the opposite is true, then some extension on u_{n+1} must be made for the time interval $(t_1^{(n)}, t_1^{(n+1)})$ such as

$$u_{n+1}(t) = u_n(t_1^{(n)})$$

or some convenient extrapolation based on $u_n(t_1^{(n)})$ and the rate of change of u_n near $t_1^{(n)}$. When the estimated solution is near the optimum, signify by relatively small values of $\nabla E(u_n)$, a more accurate determination for the entering time and the entering corner being

‡ $t_1^{(m)}$ denotes t_1 for the mth iteration.

desirable (also the terminal time and terminal state). Some refinement in step size "dt" in solving the differential equation near t_1 is necessary in order to minimize rounding errors.

Concerning the exit time, as it was observed by McIntyre and Paiewonsky [16], the conditions for leaving the constraint surface cannot be translated into mathematical statements that can be used in a computing process. Again, t_2 must be estimated and an improvement made in the estimation according to some means such as to increase t_2 when the new control causes the trajectory to violate the constraint surface and decrease t_2 otherwise. The amount of suitable change involved depends on the problem at hand. Often too large a change may cause the trajectory and some subsequent trajectories to deviate greatly from the optimal, while making too small changes would waste unnecessary computing time.

5.9 Determination of Optimum α_n

It is convenient to divide the state variable constraint problem into three parts in the following discussions, and designate them as Region I, for $t \varepsilon [t_o, t_1)$; Region II, for $t \varepsilon [t_1, t_2]$; and Region III, for $t \varepsilon (t_2, t_f]$. For Region II, the computation for the optimum α_n, the value of α that minimizes the performance functional, or step size in the search, is not involved since the control on the boundary arc is not free to vary. For Region III, the optimum α_n may be determined by using Equation (3.10) as a guide for the search and to compute the performance functional for selected values of α. A quadratic interpolation may be employed to improve the effectiveness of the search and to reduce the number of values of α needed to be considered. The computation for the optimum α_n for Region I needs further attention. First of all, due to the presence of the constraint conditions (5.6) and (5.7), the values of α to be considered must be selected in such a way that $u_n + \alpha p_n$ are admissible controls. This is a one-dimensional minimization problem subject to some side conditions. It is desirable to limit the number of values of α to be considered so that the computational time is reasonable while maintaining a tolerable accuracy on the approximating solution for each iteration. Secondly, the evaluation of the performance functional is not as simple as in the case for Region III since the value of $\phi[x(t_f)]$ will not be known until the complete trajectory is computed which includes Region III where the trajectory is as yet to be evaluated. Some equivalent condition at $t < t_f$ instead of $\phi[x(t_f)]$ sometimes may be used as in the re-entry vehicle problem below, or as in the minimum time problem selecting the optimum α_n to be the one that is nearest to the value of α Equation (3.10) provided, while satisfying the condition that $u_n + \alpha_n p_n$ is an admissible control.

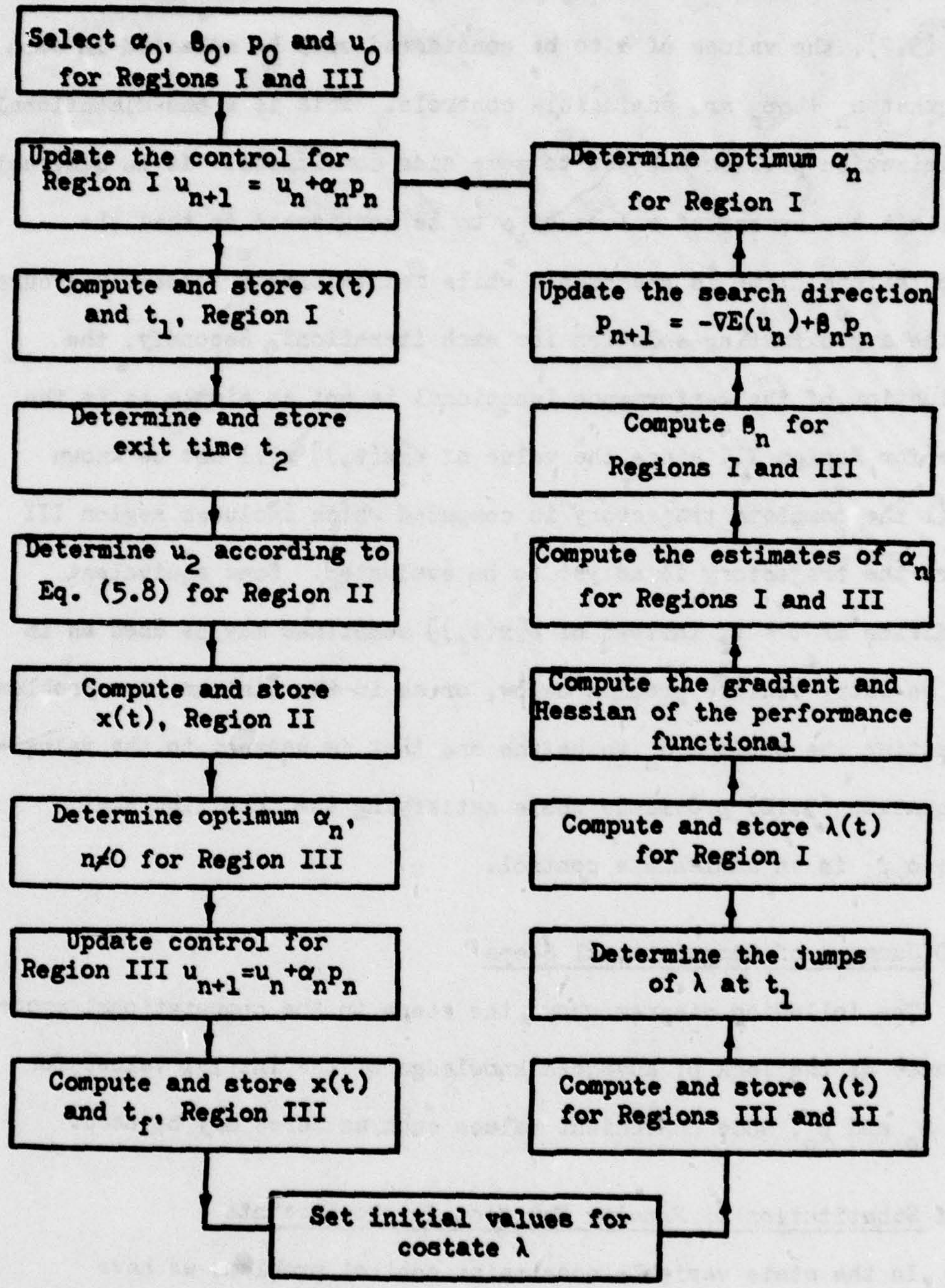
5.10 Summary of Computational Steps

The following diagram shows the steps in the computational process. Because of the lack of advanced knowledge of the initial values for α_o, β_o and p_o, some convenient values such as zeros may be used.

5.11 Substitution of Penalty Function for Constraints

In the state variable constraint control problems we have discussed above, most of the hardships in computation arise from the

Figure 5. Flow diagram for the computational Process



constraint requirements (5.6) and (5.7). The penalty function technique is designed to alleviate these difficulties. Instead of attacking the problem directly, it reformulates the control problem with state variable constraint into an unconstrained problem wherein the original performance functional is augmented by a non-negative penalty term, a function of the state variable x which increased in value with trajectories that violate the state variable constraints. By selecting a suitable sequence of non-negative penalty functions in the iteration process, it is conceivable that in many cases the desired solution for the original problem would be achieved as the limit of the sequence of approximating solutions obtained in the iteration. Indeed, this technique has been given rigorous justifications by various investigators, Moser [17], Russell [18] and Okamura [19], just to mention a few. For most penalty functions the intermediate trajectories usually violate the constraints. That is, some portion of the boundary arc is approached from outside of the constraint set.

An adaptation of this technique to suit the conjugate gradient computational method is as follows. A new performance functional is given by

$$E'(u_n) = E(u_n) + \int_{C_n} \pi(x,n)ds$$

$$= \phi[x(t_f)] + \int_{C_n} \pi(x,n)ds \qquad (5.55)$$

where C_n is the trajectory under the control u_n and the non-negative penalty function π as a function of x has the properties that for x within the constraint set π has small values relative to $\phi[x(t_f)]$, and for x outside of the constraint set π has large values that increase with the distance (with some suitable metric) away from the constraint surfaces. And as a function of n for a given x, π is a monotonically increasing function for x outside of the constraint set and conversely for x within the constraint set. The gradient of E' in general does not approach zero as the optimum solution is near, due to the added penalty term $\int_{C_n} \pi(x,n)ds$. Therefore, some other means must be used to signal that the optimum solution is near in the iteration process. Comparison of the values for $\phi[x(t_f)]$ in consecutive iterations often fail whenever the performance functional has a "flat bottom" feature. Often direct comparison of u_{n+k} with u_n is necessary, such as evaluating the quantity

$$||u_{n+k}-u_n||^2 = \sum_{j=1}^{k} \alpha_{n+j-1}\langle p_{n+j-1}, p_{n+j-1}\rangle \qquad (5.56)$$

To avoid instability in computation which causes the intermediate trajectories to swing far from the optimal trajectory and may sometimes cause the approximating solutions to diverge, the penalty function cannot be too "harsh." On the contrary, the solution may have a very slow convergence rate which would make the computation inefficient. Some compromise must be made so that each iteration brings the approximating solution closer and closer to the optimum at some reasonable rate. After the selection of the penalty function, the computational steps are the same as the one given above in Figure 5.4

for the constraint problem except for the removal of the blocks concerning Regions II and III plus some obvious modifications.

SECTION VI

A MINIMUM DISTANCE WITH FORBIDDEN REGION PROBLEM

6.1 Problem Description

As an application of the foregoing discussion to the state variable constraint problems, let us now consider a problem of moderate computational difficulty so that the features of the conjugate gradient method can be observed with greater clarity. Suppose that among the planar curves joining the point (4,1/4) and some point on the parabola with its vertex at the origin while avoiding a circular region as shown in Figure 6.1, it is desired to find one that minimize the length of the curve. The control version of this problem would be to find the time-optimal control for a piecewise smooth path satisfying the specified conditions traversed at a constant speed, where the control

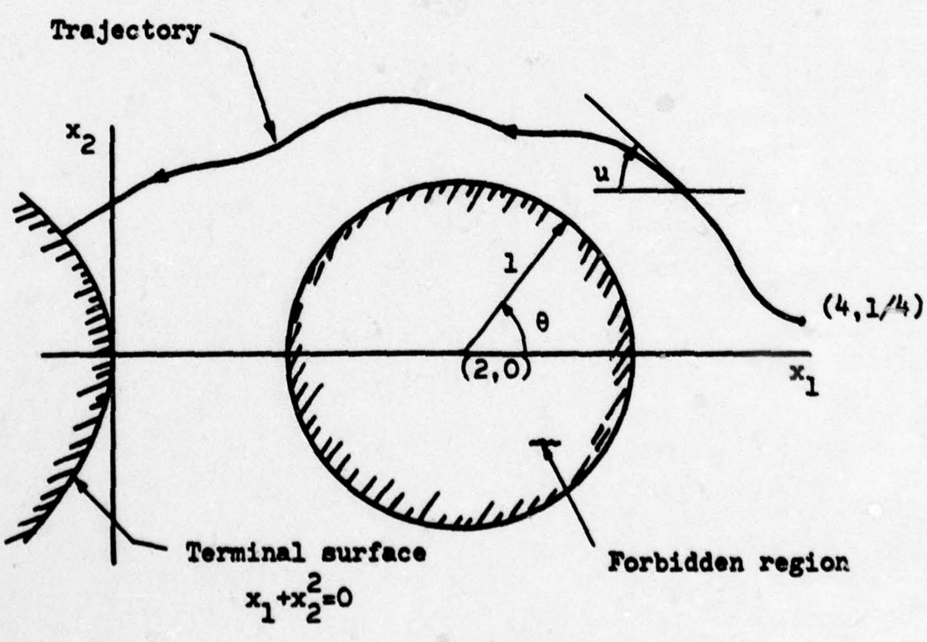


Figure 6.1. Geometry of the minimum distance problem

variable u is taken as the angle formed by the tangent to the path and the negative x_1-axis (see Figure 6.1).

This problem will be solved using the conjugate gradient method two ways. First, the computation will be carried out considering the constraints directly and then employing a penalty function to convert the constraint problem to an equivalent unconstrained one. Finally, another computational technique, the popular steepest descent, is studied with the same considerations given as in the first case of the conjugate gradient method.

The performance functional to be minimized is

$$
\begin{aligned}
E(u) &= \phi[x(t_f)] \\
&= -x_3(t_f) \\
&= -\int_{t_o}^{t_f} dt \qquad (6.1)
\end{aligned}
$$

The system dynamics can be written as

$$
\begin{aligned}
\frac{dx_1}{dt} &= -k \cos u \\
\frac{dx_2}{dt} &= k \sin u \qquad (6.2) \\
\frac{dx_3}{dt} &= 1
\end{aligned}
$$

where the constant k will be taken as unity in the sequel for simplicity. Letting $t_o = 0$, the initial conditions are

$$x_1(0) = 4$$

$$x_2(0) = 1/4 \tag{6.3}$$

$$x_3(0) = 0$$

6.2 Formulation for Numerical Computation Using Constraints Directly

The Hamiltonian associated with this problem is

$$H(x,\lambda,u) = -\lambda_1 \cos u + \lambda_2 \sin u + \lambda_3 \tag{6.4}$$

and therefore the costate equation along the interior arcs, or Regions I and III, is

$$\frac{d\lambda}{dt} = 0 \tag{6.5}$$

In view of Equation (5.34) and that $\phi(x) = -x_3$ and $\psi(x) = x_1+x_2^2$, at the terminal time t_f

$$\lambda_1(t_f) = 1$$

$$\lambda_2(t_f) = 2x_2(t_f) \tag{6.6}$$

$$\lambda_3(t_f) = -1$$

According to Equations (5.36) and (5.53), we have for the gradient and Hessian of the performance functional, respectively,

$$\nabla E(u) = \lambda_1 \sin u + \lambda_2 \cos u \tag{6.7}$$

$$F(u) = \lambda_1 \cos u - \lambda_2 \sin u \tag{6.8}$$

On the boundary arc, or Region II, the control is required to maintain the trajectory so that it will lie on the circle $(x_1-2)^2 + x_2^2 = 1$, hence

$$u(t) = \cos^{-1} x_2(t)$$

or

$$u(t) = \sin^{-1}[x_1(t)-2] \qquad (6.9)$$

Using Equation (5.28), the costate equations for Region II are

$$\frac{d\lambda_1}{dt} = -\cos u(\lambda_1 \sin u + \lambda_2 \cos u)$$

$$\frac{d\lambda_2}{dt} = \sin u(\lambda_1 \sin u + \lambda_2 \cos u) \qquad (6.10)$$

$$\frac{d\lambda_3}{dt} = 0$$

By Equations (5.30) and (5.31), the jumps of the costate at the entering corner are governed by

$$\lambda_1(t_1^-) = \lambda_1(t_1^+) + \mu\, 2[x_1(t_1^+)-2]$$

$$\lambda_2(t_1^-) = \lambda_2(t_1^+) + \mu 2x_2(t_1^+) \qquad (6.11)$$

$$\lambda_3(t_1^-) = \lambda_3(t_1^+)$$

$$\lambda_1(t_1^-)\frac{dx_1(t_1^-)}{dt} + \lambda_2(t_1^-)\frac{dx_2(t_1^-)}{dt} = \lambda_1(t_1^+)\frac{dx_1(t_1^+)}{dt} + \lambda_2(t_1^+)\frac{dx_2(t_1^+)}{dt}$$

and from which

$$\mu = \frac{\lambda_1(t_1^+)[-x_2(t_1)+\cos u(t_1^-)] + \lambda_2(t_1^+)[x_1(t_1)-2-\sin u(t_1^-)]}{2\{[x_1(t_1)-2]\cos u(t_1^-)-x_2(t_1)\sin u(t_1^-)\}} \qquad (6.12)$$

It is worth observing that in the iteration process, precaution must be taken to avoid overflows in computation since the denominator of (6.12) may vanish when the approximating trajectory is tangent to the

circle at t_1. When this occurs, the numerator vanishes also. In view of the limiting processes involved in Equation (5.25), we may therefore apply L'Hospital's rule to Equation (6.12) and obtain

$$\mu = \frac{1}{2}\left[-\lambda_1(t_1^+)\sin u(t_1^+) - \lambda_2(t_1^+)\cos u(t_1^+)\right]$$

After a control is chosen, and the initial conditions (6.3) are given, the differential equation (6.2) can be solved in a straightforward manner. To be able to evaluate the gradient and the Hessian of E, the costate in Regions I and III is needed. The values of the costate in Region III are clear from (6.5) and (6.6). By solving (6.10) backwards from t_2 to t_1, we have $\lambda_j(t_1^+)$, j=1,2,3. Through Equation (6.11), $\lambda_j(t_1^-)$, j=1,2,3 may be determined, and consequently the values of the costate for Region I obtained.

The set of admissible controls Q consists of controls that produce trajectories starting at (4,1/4) and terminating at some point on the parabola $x_1+x_2^2 = 0$ avoiding the region $(x_1-2)^2 + x_2^2 < 1$. If v and w are admissible controls with corresponding trajectories C_1 and C_2 so that the forbidden region is not in the interior of the region D bounded by curves C_1, C_2 and $x_1+x_2^2 = 0$, then for $0 \leq a \leq 1$, the control av+(1-a)w would be admissible since the resulting trajectory lies entirely in D. Hence Q is convex, and according to Theorem 3.8, if the control selected for the initial iteration is in the neighborhood of u* within which the Hessian of E is positive definite, the convergence of the iteration process is assured.

6.3 Formulation for Numerical Computation Using Penalty Function

Let us now examine what modifications must be made when the penalty function is introduced so that the problem with state variable constraint becomes an unconstrained problem. Let

$$\pi(x,n) = .01[(x_1-2)^2+x_2^2]^{A(n)} \tag{6.13}$$

where

$$A(n) = \begin{cases} 3+n, & 1 \le n < 10 \\ 14+2(n-10), & 10 \le n < 20 \\ 34+4(n-20), & n \ge 20 \end{cases}$$

As n becomes large, the contribution to the performance functional along the trajectory exterior to the circle is small, and $\pi(x,n)$ is positive everywhere except for one point, $x_1 = 2$ and $x_2 = 0$, which is zero. Hence π possesses the desired characteristics stated in the previous section.

The new performance functional to be minimized is

$$\begin{aligned} E(u) &= -x_3(t_f) \\ &= -\int_0^{t_f} dt + \int_0^{t_f} \pi(x,n)dt \end{aligned} \tag{6.14}$$

The equations describing the system dynamics (6.2) remain the same except for the last expression which becomes

$$\frac{dx_3}{dt} = 1 - \pi(x,n) \tag{6.15}$$

The initial conditions for the states are again given by (6.3). The new Hamiltonian is

$$H(x,\lambda,u) = -\lambda_1 \cos u + \lambda_2 \sin u + \lambda_3[1-\pi(x,n)] \tag{6.16}$$

and the costate equations are

$$\frac{d\lambda_1}{dt} = \lambda_3 \frac{\partial \pi}{\partial x_1}$$

$$\frac{d\lambda_2}{dt} = \lambda_3 \frac{\partial \pi}{\partial x_2} \tag{6.17}$$

$$\frac{d\lambda_3}{dt} = 0$$

with the terminal conditions as given by (6.6). The gradient and the Hessian are again given by (6.7) and (6.8). Since there are no constraint conditions involved, the computer program becomes considerably simpler. Any piecewise control having the corresponding trajectory satisfying the initial and terminal conditions is an admissible control, and Q is again convex.

6.4 Computational Results

Many different estimated controls have been selected for the initial iteration in the course of studies for computation as a constraint problem and as an unconstrained problem using either the conjugate gradient method or the method of steepest descent, but only one is presented below in detail for each way of solving this problem since the convergence characteristics are similar for any estimated control for which the iteration process converges.

Treating the problem as a constraint problem, the following initial estimates are used

$$u_o(t) = -\frac{1}{3}t, \qquad 0 \leq t \leq t_1$$

$$u_o(t) = .1+.05 \sin 10t, \qquad t_2 \leq t \leq t_f$$

$\theta_o = 2.8$ (see Figure 6.2)

The change in θ, $\Delta\theta$ is made according to the following:

(i) If the trajectory under the new estimated control does not violate the constraining circle in the kth iteration, then

$$\Delta\theta = \begin{cases} -.1 \quad , & k < 8 \\ -.01 \quad , & 8 \le k \le 15 \\ -.0005, & k > 15 \end{cases}$$

(ii) If the trajectory under the new estimated control violates the constraining circle as shown below, where A is the point where the trajectory leaves the circle in the (k-1)st iteration and B is the last intersection of the trajectory C_k under the new control and the circle, then

$\Delta\theta$ = length of arc AB

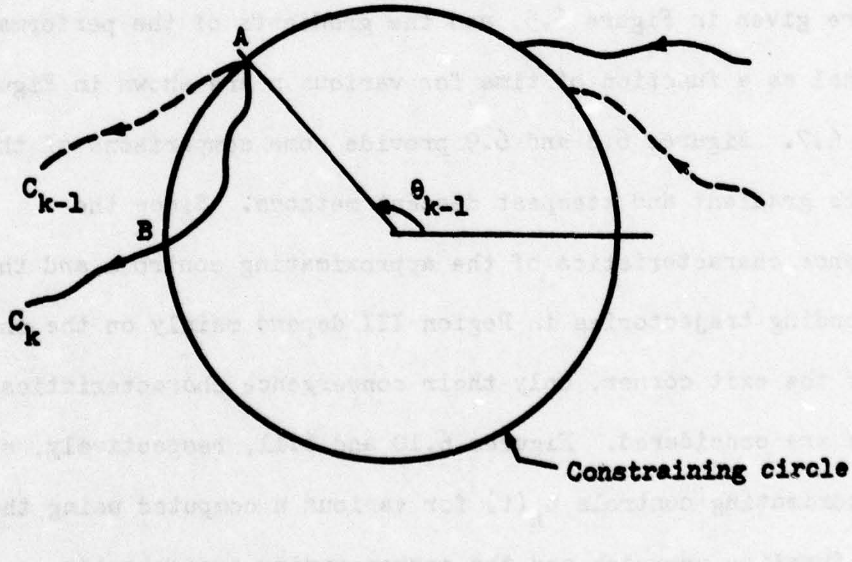


Figure 6.2. Trajectory C_k violating constraining circle

Treating the problem as an unconstrained problem, the initial estimate was

$$u_o(t) = 1 - .4t$$

The fourth order Runge-Kutta method was used in solving the differential equations with step-size dt = .01, except for the small intervals slightly before the trajectory intersects either the terminal curve $x_1+x_2^2 = 0$ or the constraining circle, where dt = .0002.

The computed results are shown in Figures 6.3 to 6.11. Comparisons are made between the different approaches of solution whenever possible. In the following, whenever there is no mention as to whether the solution is obtained by using constraints directly or by using a penalty function it is understood that the first is used. Figures 6.3 and 6.4, respectively, show the approximating controls $u_n(t)$ for various n computed by the conjugate gradient method and the corresponding trajectories. The values of the performance functional vs. n are given in Figure 6.5, and the gradients of the performance functional as a function of time for various n are shown in Figures 6.6 and 6.7. Figures 6.8 and 6.9 provide some comparisons of the conjugate gradient and steepest descent methods. Since the convergence characteristics of the approximating controls and the corresponding trajectories in Region III depend mainly on the choice of θ_n or the exit corner, only their convergence characteristics in Region I are considered. Figures 6.10 and 6.11, respectively, show the approximating controls $u_n(t)$ for various n computed using the penalty function approach and the corresponding trajectories.

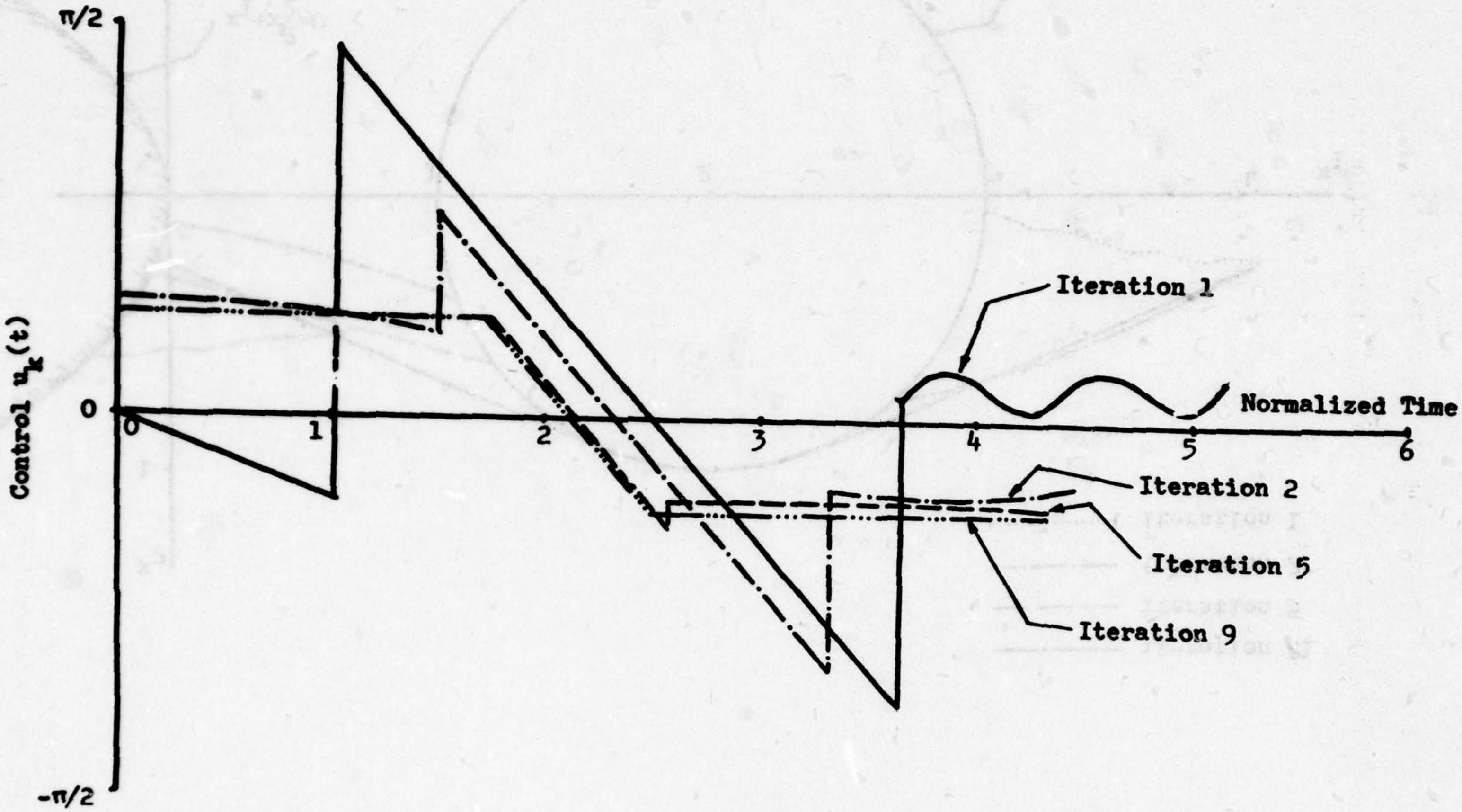


Figure 8. Approximating Controls Computed by Method of Conjugate Gradients

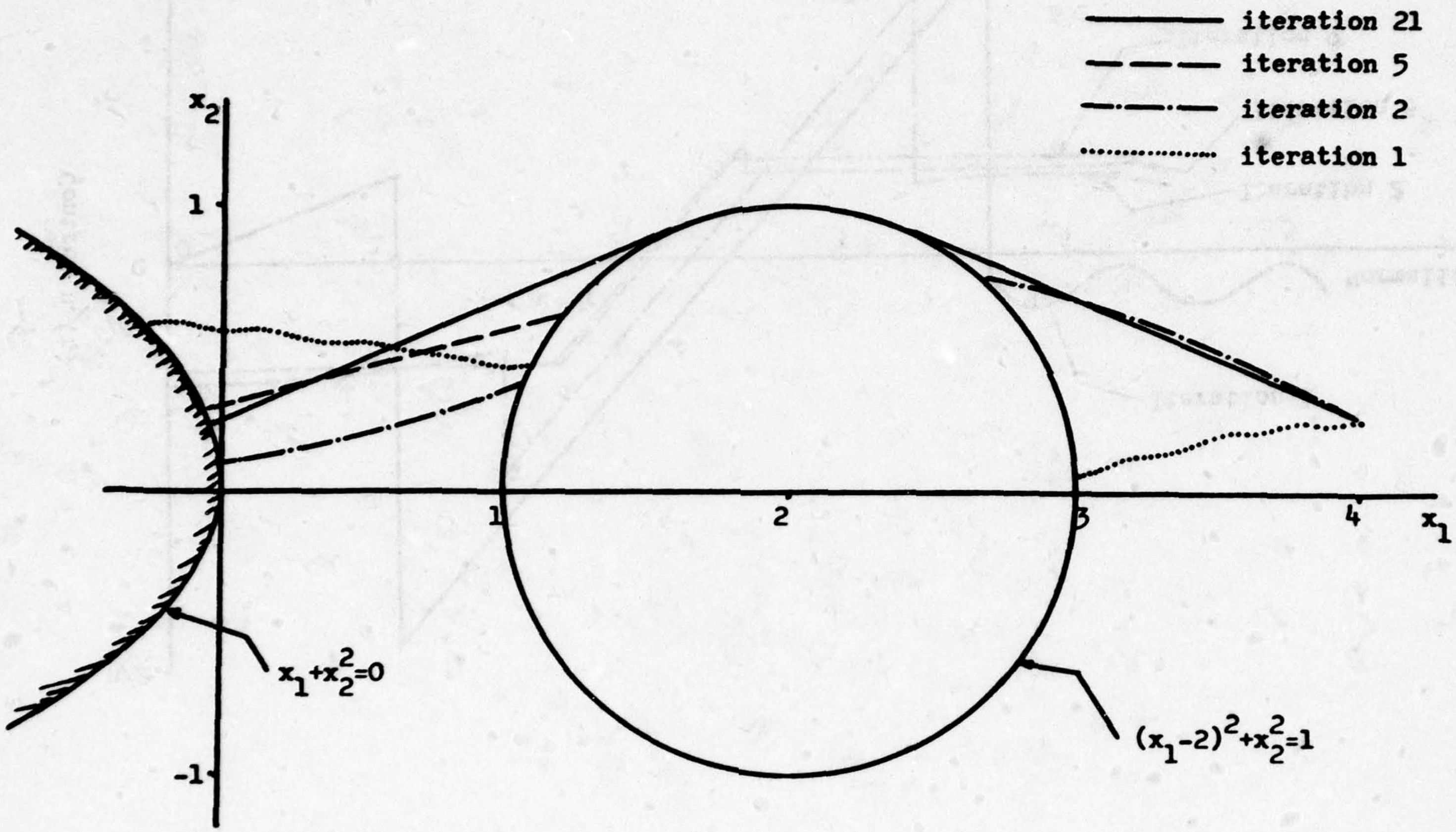


Figure 9. Trajectories Computed by Method of Conjugate Gradients

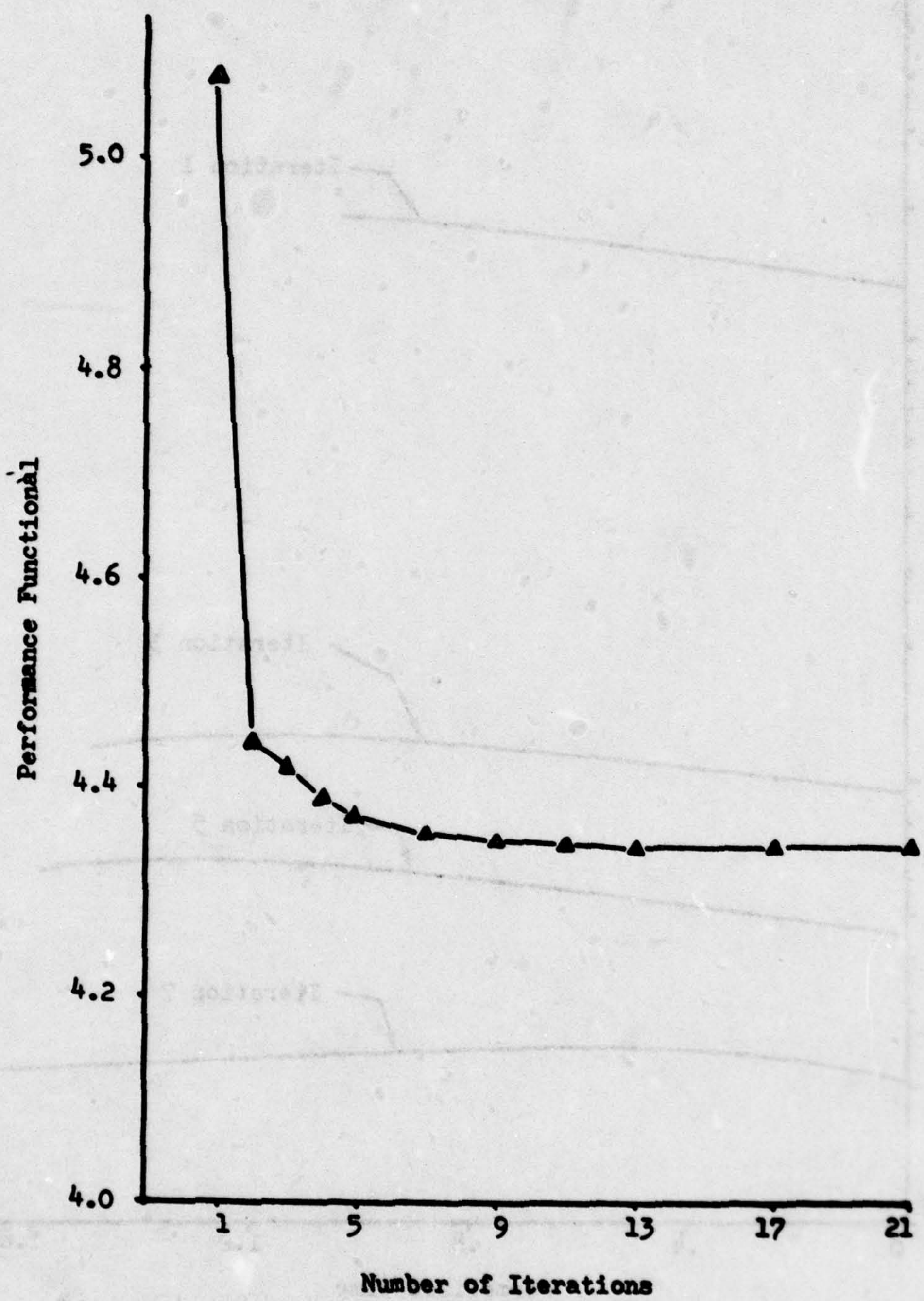
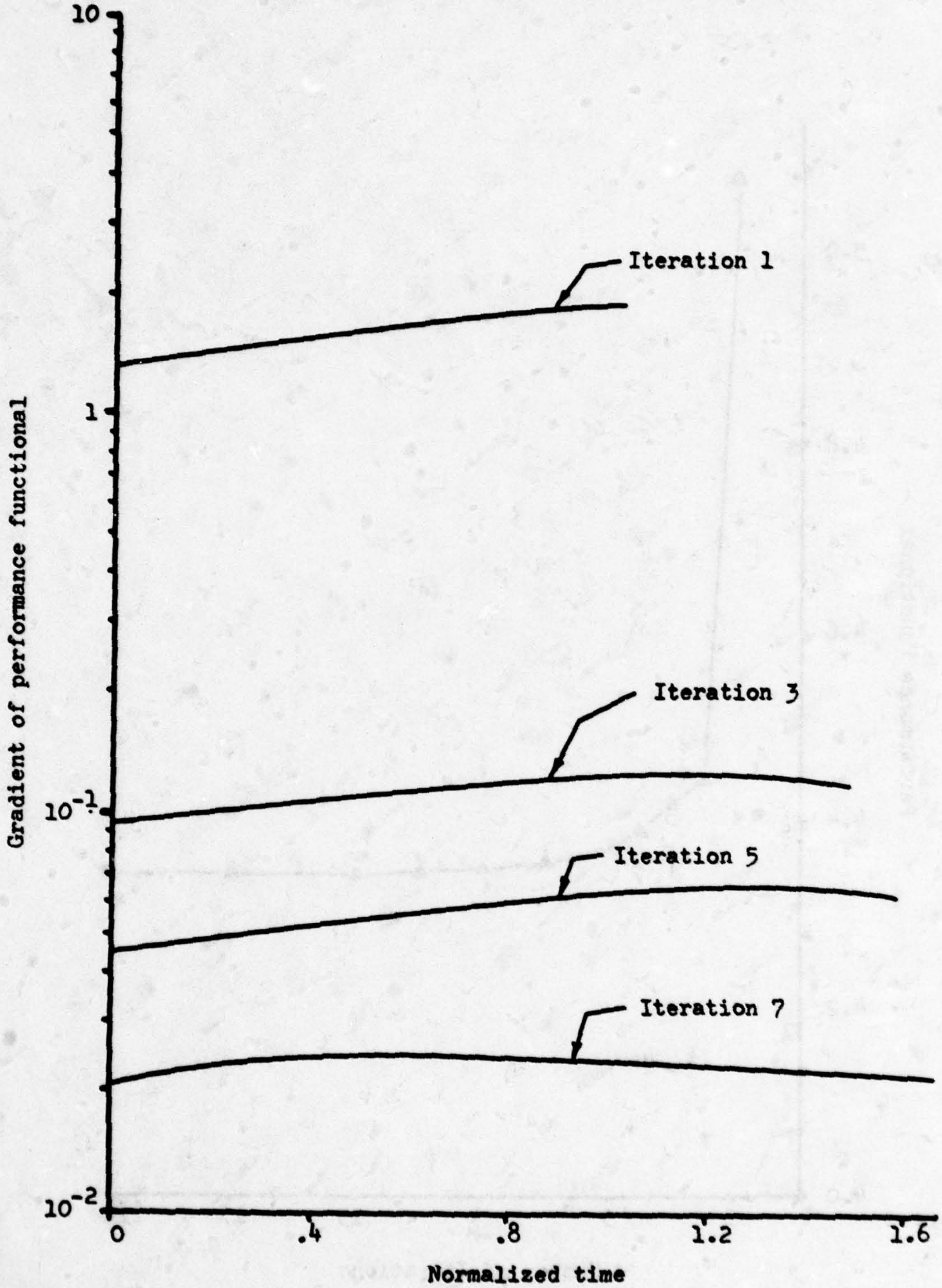


Figure 10. Performance Functional by Conjugate Gradient Method

Figure 6.6. Gradients of performance functional, Region I



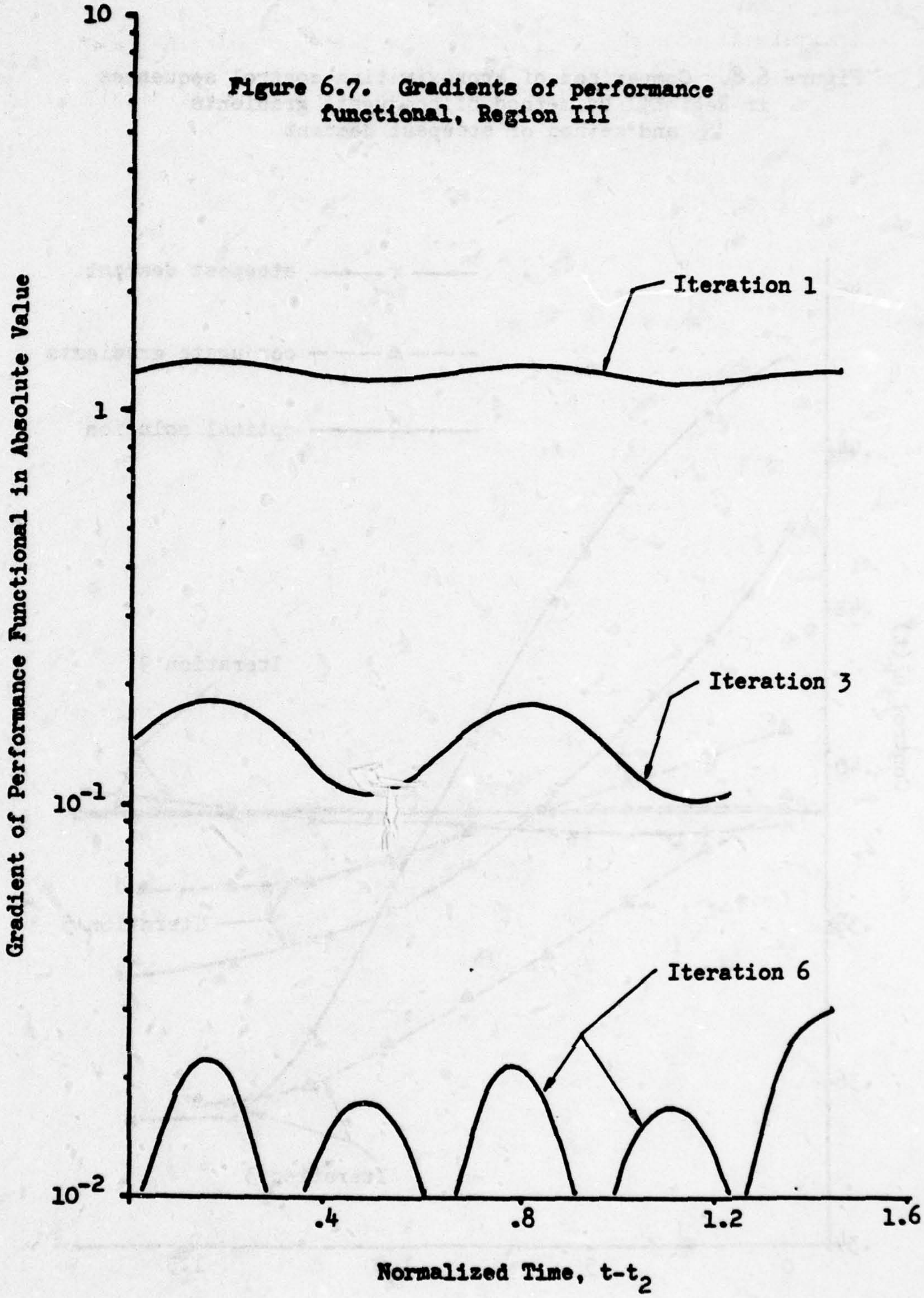


Figure 6.7. Gradients of performance functional, Region III

Figure 6.8. Comparison of approximating control sequences in Region I by method of conjugate gradients and method of steepest descent

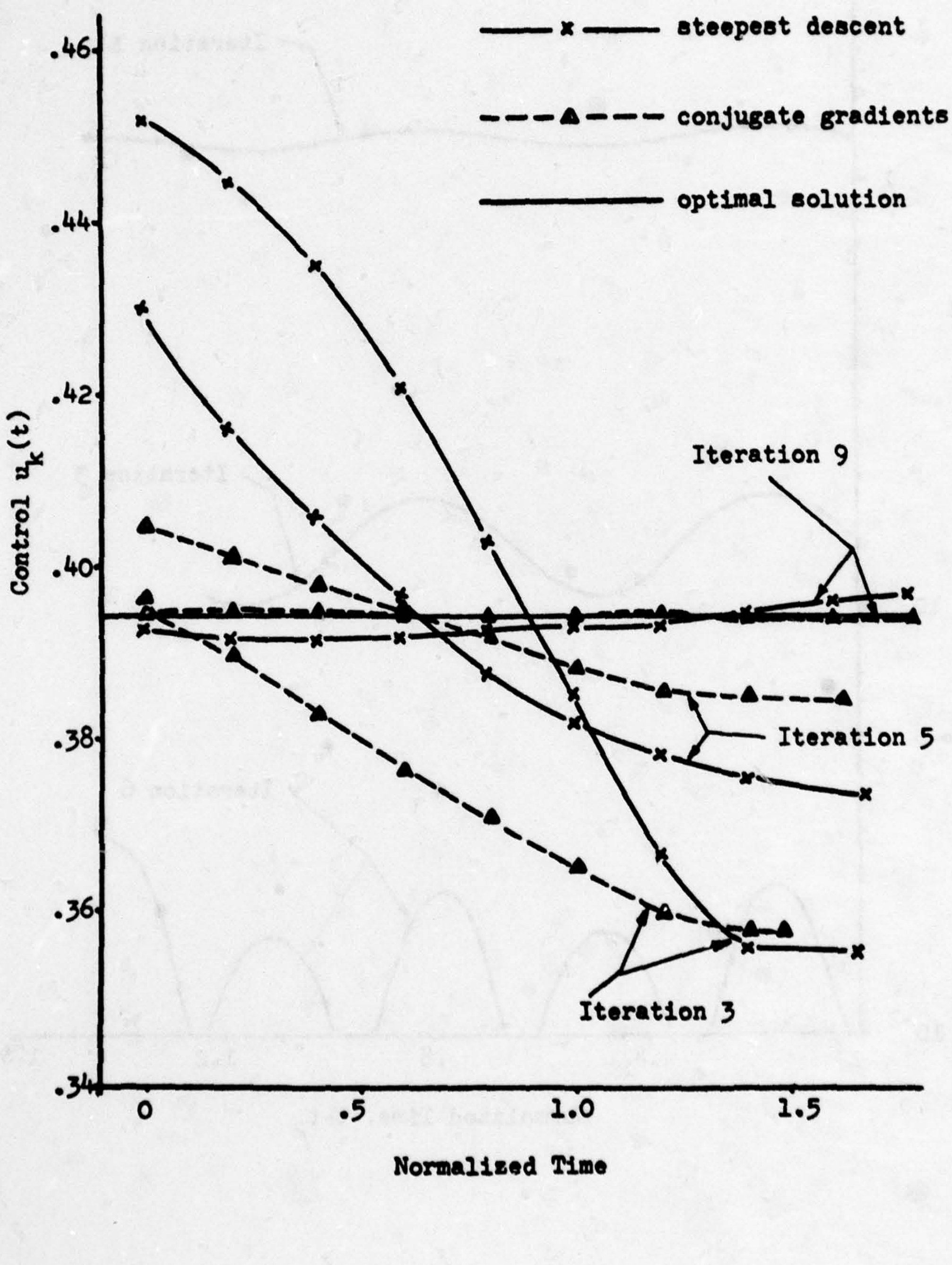


Figure 6.9. Deviation of approximating controls from optimal by method of conjugate gradients and method of steepest descent

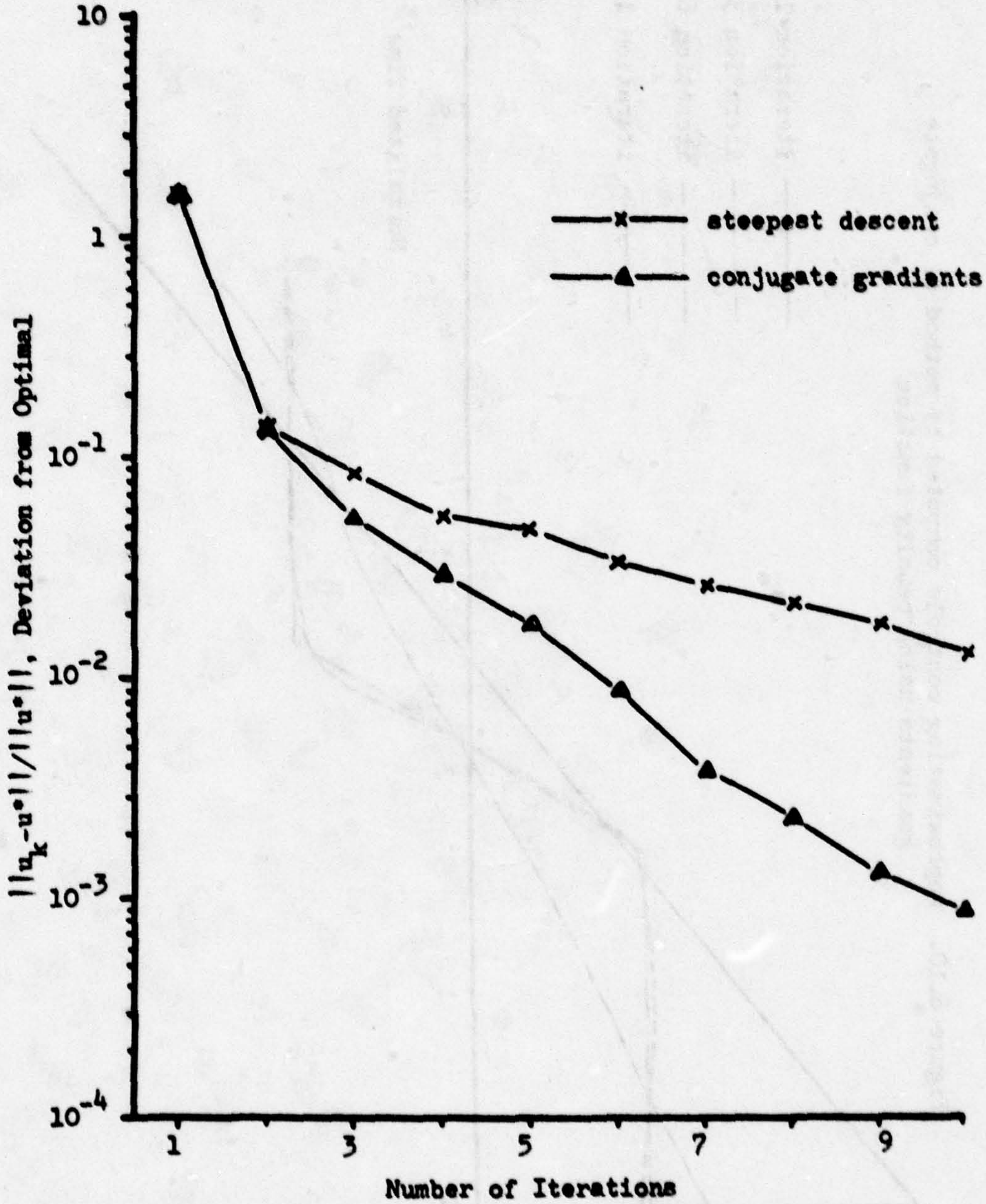


Figure 6.10. Approximating controls computed by method of conjugate gradients using penalty function

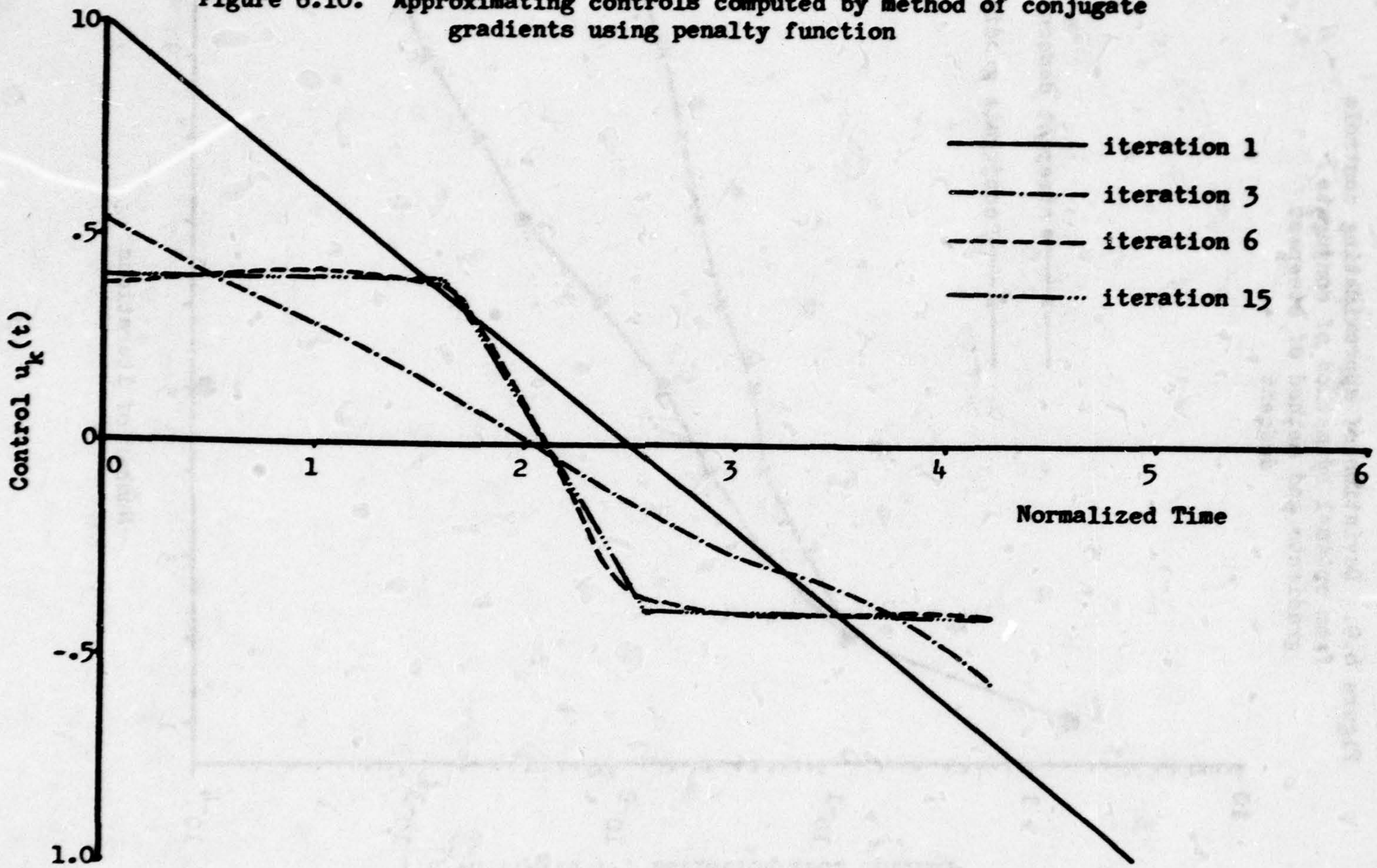
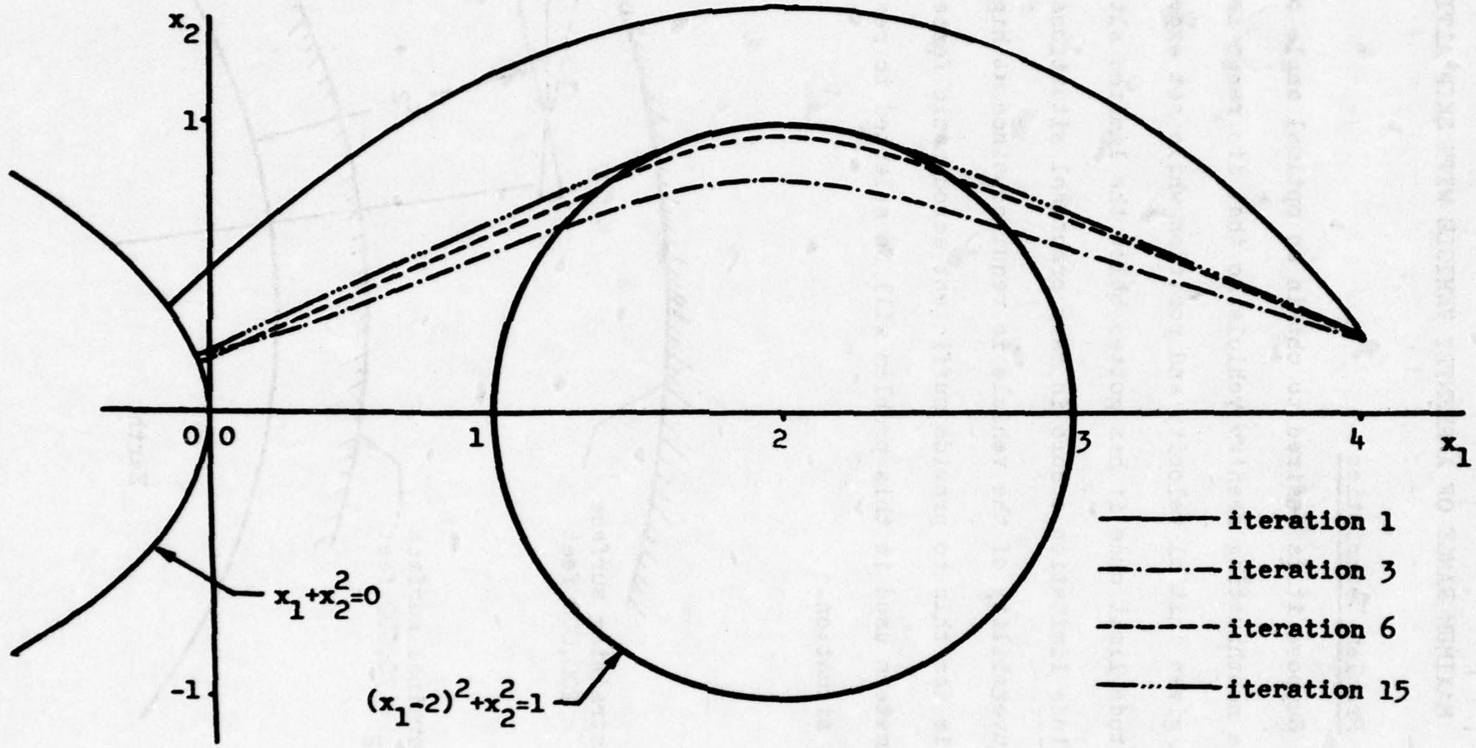


Figure 6.11. Trajectories computed by method of conjugate gradients using penalty functions



SECTION VII

MAXIMUM RANGE OF A REENTRY VEHICLE WITH SKIP ALTITUDE CONSTRAINT

7.1 Problem Description

Suppose it is desired to obtain an optimal angle of attack program for a nonthrusting reentry vehicle so that its range is maximized for some given initial velocity and position while not exceeding a certain altitude limit once it has gotten below the limited altitude. The altitude limitation occurs in many practical situations whenever maneuverability of the vehicle is required since at high altitudes the air is too thin to provide sufficient aerodynamic forces. The parameters used in this problem will be selected to resemble a typical real situation.

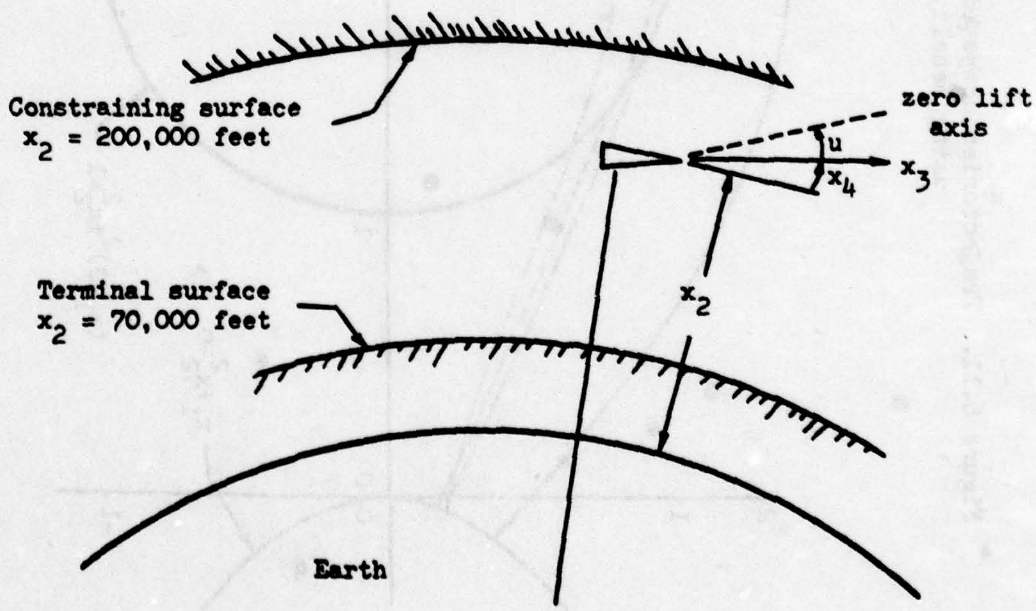


Figure 7.1. Geometry of the reentry vehicle problem

The earth will be considered as spherical and non-rotating. This state variable constraint control problem may be formulated as follows: Let x_1 be the range along the earth; x_2 be the altitude of the vehicle; x_3 be its velocity; and x_4 be the flight path angle relative to local horizontal. The constraining surface is

$$g(x) = x_2(t) - 200{,}000 \text{ feet} \tag{7.1}$$

and suppose that the terminal surface is

$$\psi(x) = 70{,}000 \text{ feet} - x_2(t) \tag{7.2}$$

The performance functional to be minimized is

$$\begin{aligned} E(u) &= \phi[x(t_f)] \\ &= -x_1(t_f) \end{aligned} \tag{7.3}$$

where t_f is the time when the vehicle is at an altitude of 70,000 feet. The motion of the vehicle is given by the following equations:

$$\begin{aligned}
\frac{dx_1}{dt} &= \frac{x_3}{1 + x_2/R} \cos x_4 \\
\frac{dx_2}{dt} &= x_3 \sin x_4 \\
\frac{dx_3}{dt} &= -(.274 + 1.8 \sin^2 u)\rho x_3^2 \frac{S}{2m} - g \sin x_4 \\
\frac{dx_4}{dt} &= .6 \sin 2u\rho x_3 \frac{S}{2m} - \frac{g}{x_3} \cos x_4 + \frac{x_3}{R+x_2} \cos x_4
\end{aligned} \tag{7.4}$$

where

g, the local gravity $= 32.2 \left(\frac{R}{R+x_2}\right)^2$

ρ, the atmospheric density = .00238 exp $(-x_2/24{,}000)$ slugs/ft^3

R, the radius of earth = 2.1×10^7 feet

S = the effective surface in ft^2

m = the mass of the vehicle in slugs

Let t_0=0 and the initial conditions be

$$\begin{aligned} x_1(0) &= 0 \text{ feet} \\ x_2(0) &= 340{,}000 \text{ feet} \\ x_3(0) &= 28{,}000 \text{ feet/second} \\ x_4(0) &= -.14 \text{ radians.} \end{aligned} \qquad (7.5)$$

7.2 Analysis for Numerical Computation

The Hamiltonian associated with this problem is

$$\begin{aligned} H(x,\lambda,u) = \lambda_1 \left[\frac{x_3}{1+x_2/R}\right] \cos x_4 + \lambda_2 x_3 \sin x_4 \\ - \lambda_3[(.274 + 1.8 \sin^2 u)\rho x_3^2 \frac{S}{2m} + g \sin x_4] \\ + \lambda_4 \left[.6 \sin 2u\rho x_3 \frac{S}{2m} - \frac{g}{x_3} \cos x_4 + \frac{x_3}{R+x_2} \cos x_4\right] \end{aligned} \qquad (7.6)$$

and therefore the costate equations along the interior arcs, or Regions I and III, are

$$\frac{d\lambda_1}{dt} = 0$$

$$\begin{aligned} \frac{d\lambda_2}{dt} = \lambda_1 x_3 \cos x_4 \frac{R}{(R+x_2)^2} - \lambda_3 [(.274 + 1.8 \sin^2 u)x_3^2 \frac{S}{2m} \rho/24{,}000 \\ + 2g \sin x_4/(R+x_2)] + \lambda_4 \Bigg[.6 \sin 2ux_3 \frac{S}{2m} \rho/24{,}000 \\ - \frac{2g \cos x_4}{x_3(R+x_2)} + \frac{x_3 \cos x_4}{(R+x_2)^2}\Bigg] \end{aligned} \qquad (7.7)$$

$$\frac{d\lambda_3}{dt} = -\lambda_1 \cos x_4/(1+x_2/R) - \lambda_2 \sin x_4 + \lambda_3(.274+1.8 \sin^2 u)\rho x_3\ S/m$$

$$- \lambda_4 \left[.6 \sin 2u\rho \frac{S}{2m} + \frac{g \cos x_4}{x_3^2} + \frac{\cos x_4}{(R+x_2)} \right]$$

$$\frac{d\lambda_4}{dt} = \lambda_1\left(\frac{x_3}{1+x_2/R}\right) \sin x_4 - \lambda_2 x_3 \cos x_4 + \lambda_3 g \cos x_4$$

$$- \lambda_4 \left[\frac{g \sin x_4}{x_3} - \frac{x_3 \sin x_4}{R+x_2} \right]$$

In view of Equation (5.34) and that $\phi(x) = -x_1$ and $\psi(x) = 70{,}000$ feet - x_2, at the terminal time t_f

$$\begin{aligned} \lambda_1(t_f) &= -1 \\ \lambda_2(t_f) &= -1 \\ \lambda_3(t_f) &= 0 \\ \lambda_4(t_f) &= 0 \end{aligned} \tag{7.8}$$

According to Equations (5.36) and (5.53) we have for the gradient and Hessian of the performance functional, respectively,

$$\nabla E(u) = -\lambda_3 \rho x_3^2 \frac{S}{2m} (1.8 \sin 2u) + \lambda_4 \rho x_3 \frac{S}{2m} (1.2 \cos 2u) \tag{7.9}$$

$$F(u) = -\lambda_3 \rho x_3^2 \frac{S}{2m} (3.6 \cos 2u) - \lambda_4 \rho x_3 \frac{S}{2m} (2.4 \sin 2u) \tag{7.10}$$

where an approximation has been made for the Hessian as discussed in Section V.

On the boundary arc, or Region II, according to (5.4), we have

$$g(x) = x_2 - 200{,}000 \text{ feet} = 0 \tag{7.11}$$

$$\frac{dg(x)}{dt} = x_3 \sin x_4 = 0 \tag{7.12}$$

$$\frac{d^2 g(x)}{dt^2} = \frac{dx_3}{dt} \sin x_4 + x_3 \cos x_4 \frac{dx_4}{dt} = 0 \tag{7.13}$$

Equation (7.12) implies that $x_4 = 0$, and consequently, it follows from Equation (7.13),

$$\frac{dx_4}{dt} = .6 \sin 2u\rho x_3 \frac{S}{2m} - \frac{g}{x_3} \cos x_4 + \frac{x_3}{R+x_2} \cos x_4 \Big|_{x_4=0} = 0 \tag{7.14}$$

Therefore, the control along the boundary arc is given by

$$u = \frac{1}{2} \sin^{-1} \left\{ \left[g - \frac{x_3^2}{R+x_2} \right] / .6\rho x_3^2 \frac{S}{2m} \right\} \tag{7.15}$$

and the motion is given by

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{x_3}{1+x_2/R} \\ \frac{dx_2}{dt} &= 0 \\ \frac{dx_3}{dt} &= -(.274 + 1.8 \sin^2 u)\rho x_3^2 \frac{S}{2m} \\ \frac{dx_4}{dt} &= 0 \end{aligned} \tag{7.16}$$

It follows from Equation (5.28) that the costate equations for Region II are

$$\frac{d\lambda_1}{dt} = 0$$

$$\frac{d\lambda_2}{dt} = \lambda_1 x_3 \frac{R}{(R+x_2)^2} - \lambda_3[(.274 + 1.8 \sin^2 u)x_3^2 \frac{S}{2m} \rho/24{,}000]$$

$$- \lambda_4 \left[.6 \sin 2u x_3 \frac{S}{2m} \rho/24{,}000 - \frac{2g}{x_3(R+x_2)} + \frac{x_3}{(R+x_2)^2} \right]$$

$$+ (\lambda_3\ 1.5 \tan 2u - \lambda_4/x_3)[x_3^2 \frac{S}{2m} \rho\ .6 \sin 2u/24{,}000 \qquad (7.17)$$

$$+ x_3^2/(R+x_2)^2 - 2g/(R+x_2)]$$

$$\frac{d\lambda_3}{dt} = -\lambda_1/(1+x_2/R) + \lambda_3(.274 + 1.8 \sin^2 u)\rho x_3 \frac{S}{2m}$$

$$- \lambda_4[.6 \sin 2u\rho \frac{S}{2m} + g/x_3^2 + 1/(R+x_2)]$$

$$-(\lambda_3\ 1.5 \tan 2u - \lambda_4/x_3)\left[1.2 \sin 2u\rho x_3 \frac{S}{2m} + \frac{2x_3}{R+x_2}\right]$$

$$\frac{d\lambda_4}{dt} = -\lambda_2 x_3 + \lambda_3 g$$

By Equations (5.30) and (5.31) and the fact that $x_4(t_1^+) = 0$, the jumps of the costate at the entering corner are governed by

$$\lambda_1(t_1^-) = \lambda_1(t_1^+)$$

$$\lambda_2(t_1^-) = \mu_o + \lambda_2(t_1^+)$$

$$\lambda_3(t_1^-) = \lambda_3(t_1^+) \qquad (7.18)$$

$$\lambda_4(t_1^-) = \mu_1 x_3 + \lambda_4(t_1^+)$$

$$\lambda_3 \frac{dx_3}{dt}\bigg|_{t_1^-} + \lambda_4 \frac{dx_4}{dt}\bigg|_{t_1^-} = \lambda_3 \frac{dx_3}{dt}\bigg|_{t_1^+} + \lambda_4 \frac{dx_4}{dt}\bigg|_{t_1^+}$$

It follows from the last two expressions that

$$\mu_1 = \frac{\lambda_3(t_1)\left[\left.\frac{dx_3}{dt}\right|_{t_1^+} - \left.\frac{dx_3}{dt}\right|_{t_1^-}\right]}{x_3(t_1)\left.\frac{dx_4}{dt}\right|_{t_1^-}} - \frac{\lambda_4(t_1^-)}{x_3(t_1)} \tag{7.19}$$

μ_0 as a function of the state and costate vectors is not explicit, and a trial-and-error procedure such as the one described in Section V will be used. From the physics of the problem, it is clear that in order to minimize the drag force (the first term on the right side of Equation (7.4)), and consequently the greater range the vehicle would travel, it is desirable for the vehicle to make its flight at the limit altitude of 200,000 feet for some period of time, provided that it has sufficient energy to return to that altitude after it has been below 200,000 feet once. This fact will serve as a guide in selecting the initial estimate of the control in Region I.

The continuity property of the state variable implies that $x_4(t_1^-) = 0$ which in turn implies that the admissible controls must have their corresponding trajectories tangent to the constraining surface at the entering corner. Consequently, the set of admissible controls Q does not possess the same property as the problem considered in Section VI, where the admissible set of controls is convex. Moreover, it is true that every neighborhood of an admissible control has at least one non-admissible control. We can justify this statement heuristically as follows: If u is an admissible control, then we can select a control v so that $v(t) = u(t)$ for $t \in [0,t_1-\delta]$,

$\delta > 0$, and for $t > t_1 - \delta$ select v so that the corresponding trajectory is not tangent to the constraining surface (i.e., v is not an admissible control). Since δ is arbitrary, $||u-v||$ may be made arbitrarily small.

From the computational viewpoint it is sometimes advantageous, such as in this problem, to relax the requirement that the estimated control u_k for each iteration be admissible. In other words, computational time may be considerably reduced if the optimal control u^* is approached along a "path" whose intermediate "points" may be nonadmissible conceptually as shown in Figure 7.2.

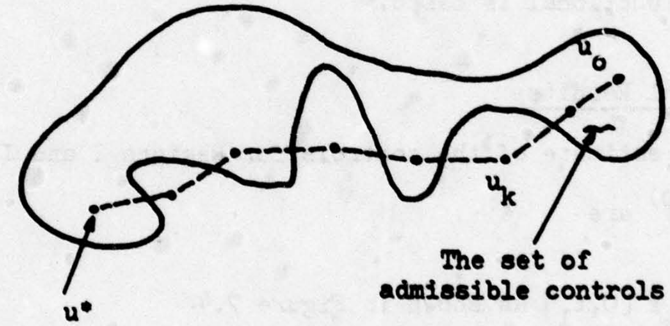


Figure 7.2. Illustration of the path leading to optimal control u^*

Let us now make this statement specific. Let the modified entering time t_1 be the smallest t that satisfies either one of the following two conditions

(i) $x_2(t) = 200{,}000$ feet

(ii) $x_4(t) = 0$ and $\dfrac{dx_4}{dt} = 0$ §

§ The flight path angle vanishes also before the pull up.

The adaptation of these two conditions to the computer program is simple. Denote the set of admissible controls for the <u>kth</u> iteration by Q_k, and let

$$Q_k = \left\{ u : x_4(t_1) < \frac{.1}{k} \text{ and } 200{,}000 - x_2(t_1) < 5{,}000/k \right\} \qquad (7.20)$$

The sequence of admissible control sets as defined above approaches Q as $k \to \infty$.

It is worthy to note that in Region I the problem of determining α that minimizes the performance functional is equivalent to the problem of maximizing x_3 for a fixed x_1 in Region II, thereby eliminating the computation of the trajectory in Region III on which the performance functional is based.

7.3 Computational Results

The initial estimate of the controls for Regions I and III and the exit time $t_2^{(0)}$ are

$u_o(t)$ for $t \in [0, t_1]$ as shown in Figure 7.4

$u_o(t) = .42$ radians, $t \in [t_2, t_f]$

$t_2^{(0)} = 235$ seconds.

The change in t_2, Δt is made according to the following:

(i) If the trajectory under the new estimated control does not violate the constraining surface $x_2 - 200{,}000 = 0$ in the <u>kth</u> iteration, then

$$\Delta t = \begin{cases} -2 & k < 4 \\ -1 & 5 \leq k \leq 8 \\ -.5 & k > 8 \end{cases}$$

(ii) If the trajectory under the new estimated control violates the constraining surface as shown below where A is the exit corner in the (k-1)st iteration and B is the last intersection of the trajectory C_k under the new control and the constraining surface, then

$$\Delta t = \frac{\text{distance between A and B}}{x_3[t_2^{(k)}]}$$

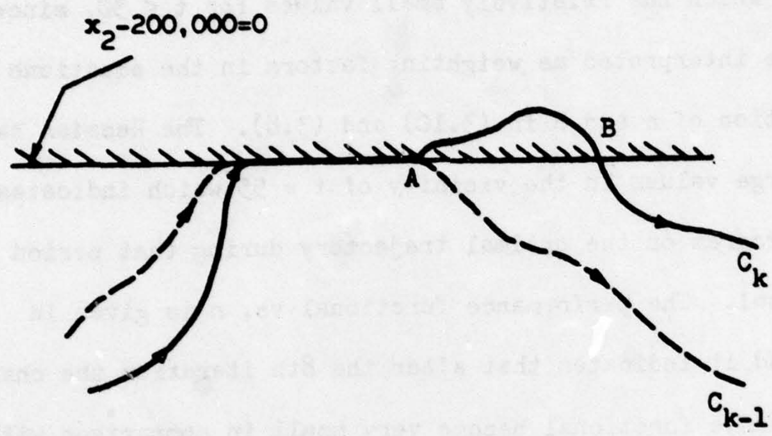


Figure 7.3. Trajectory C_k violating constraining surface

The fourth order Runge-Kutta method was used in solving the state and costate differential equations with step sizes dt = 1 for Regions II and III and dt = .5 for Region I, where greater accuracy was demanded, except for the small intervals slightly before the trajectory intersects either the terminal surface or the constraining surface where dt = .01.

Figures 7.4 and 7.5, respectively, show the approximating controls $u_n(t)$ for various n computed by the conjugate gradient method for Region I and Region III. As expected from physical consideration, the control program for the vehicle when it is above

220,000 feet has only minute influence in the trajectory since the air density is extremely small. During the ten iterations, no significant changes occurred in $u_k(t)$ for the first 30 seconds, which corresponds to x_2 greater than 220,000 feet, consequently the curve representing $u_{10}(t)$ for $t < 30$ may contain considerable amount of uncertainty. This fact may be seen by examining the Hessian of the performance functional in Figure 7.7 which has relatively small values for $t < 30$, since the Hessian may be interpreted as weighting factors in the equations for the determination of α and β in (3.10) and (3.8). The Hessian has relatively large values in the vicinity of $t = 55$ which indicates that the control program on the optimal trajectory during that period of time is critical. The performance functional vs. n is given in Figure 7.6, and it indicates that after the 8th iteration the changes in the performance functional become very small in comparison with the changes occurred in the earlier iterations. The computed results for all the states after ten iterations are presented in Tables 7.1, 7.2, and 7.3, and the costates are shown in Figure 7.8. The computed results show a very close agreement with the optimality relationship that

$$u^*(t) = \frac{1}{2} \tan^{-1}\left(\frac{2\lambda_4}{3x_3\lambda_3}\right)$$

as required by the Weierstrass condition.

Figure 7.5. The approximating controls, Region III

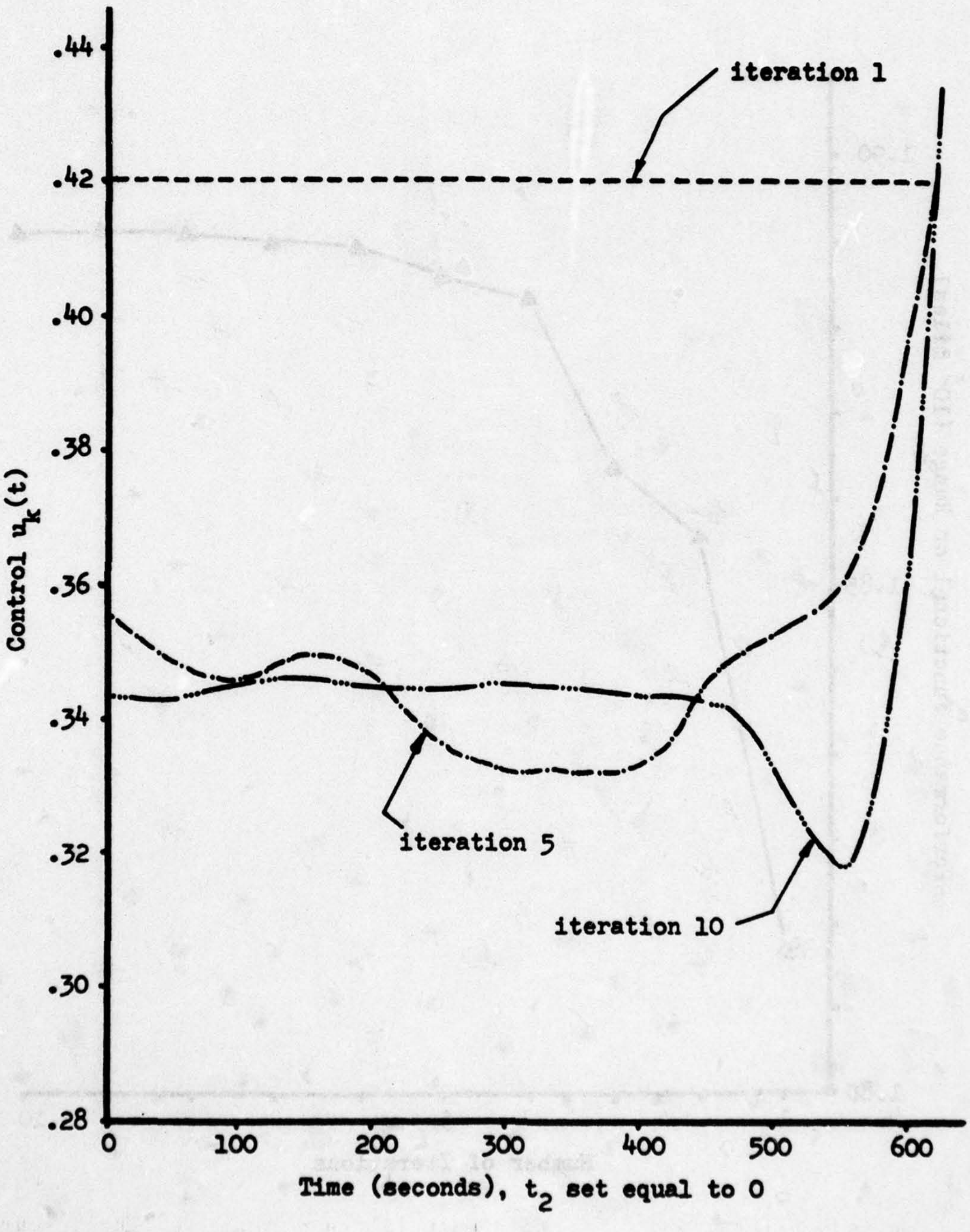


Figure 7.6. Performance functional vs number of iterations

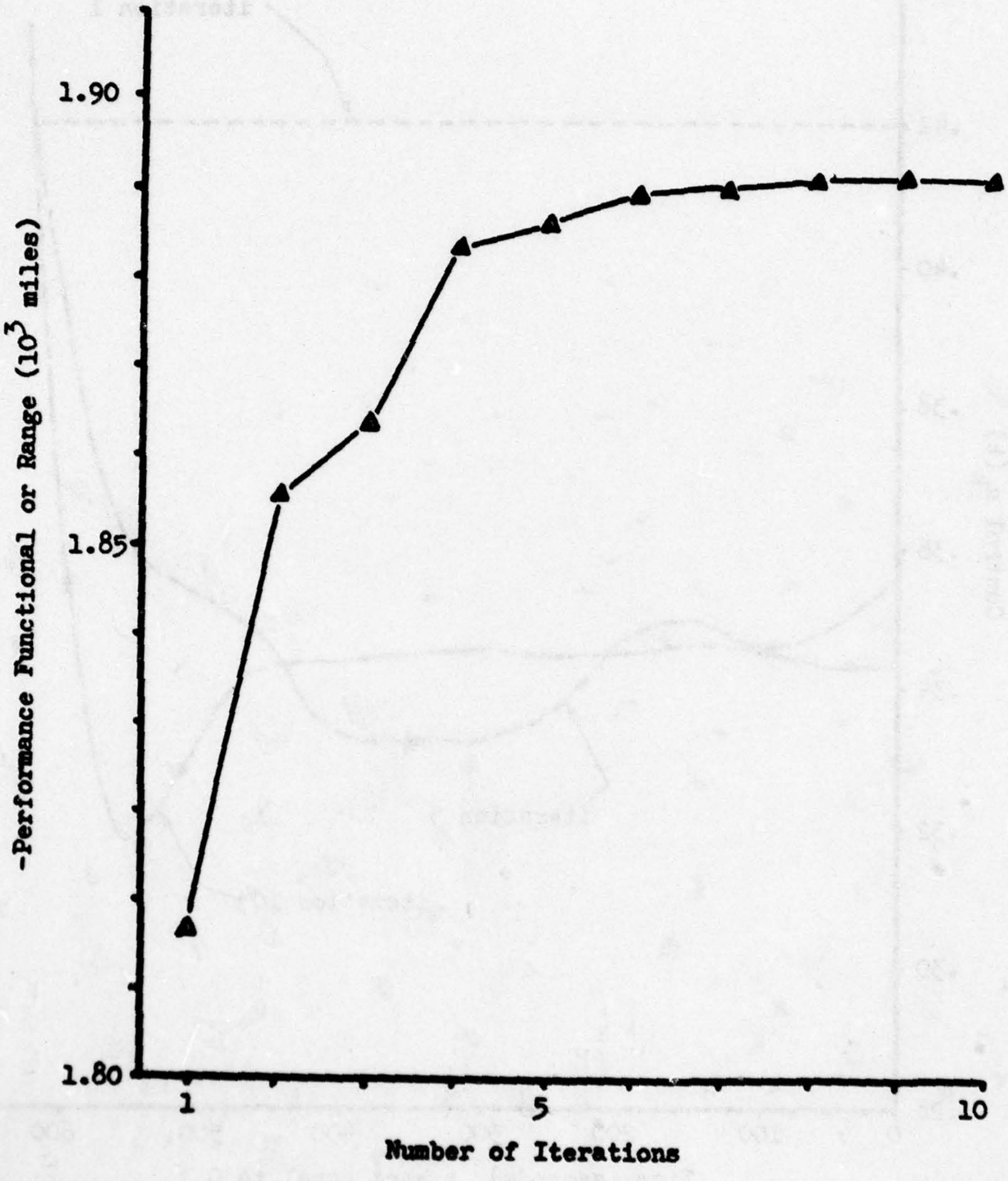


Figure 7.7. The Hessian of the performance functional

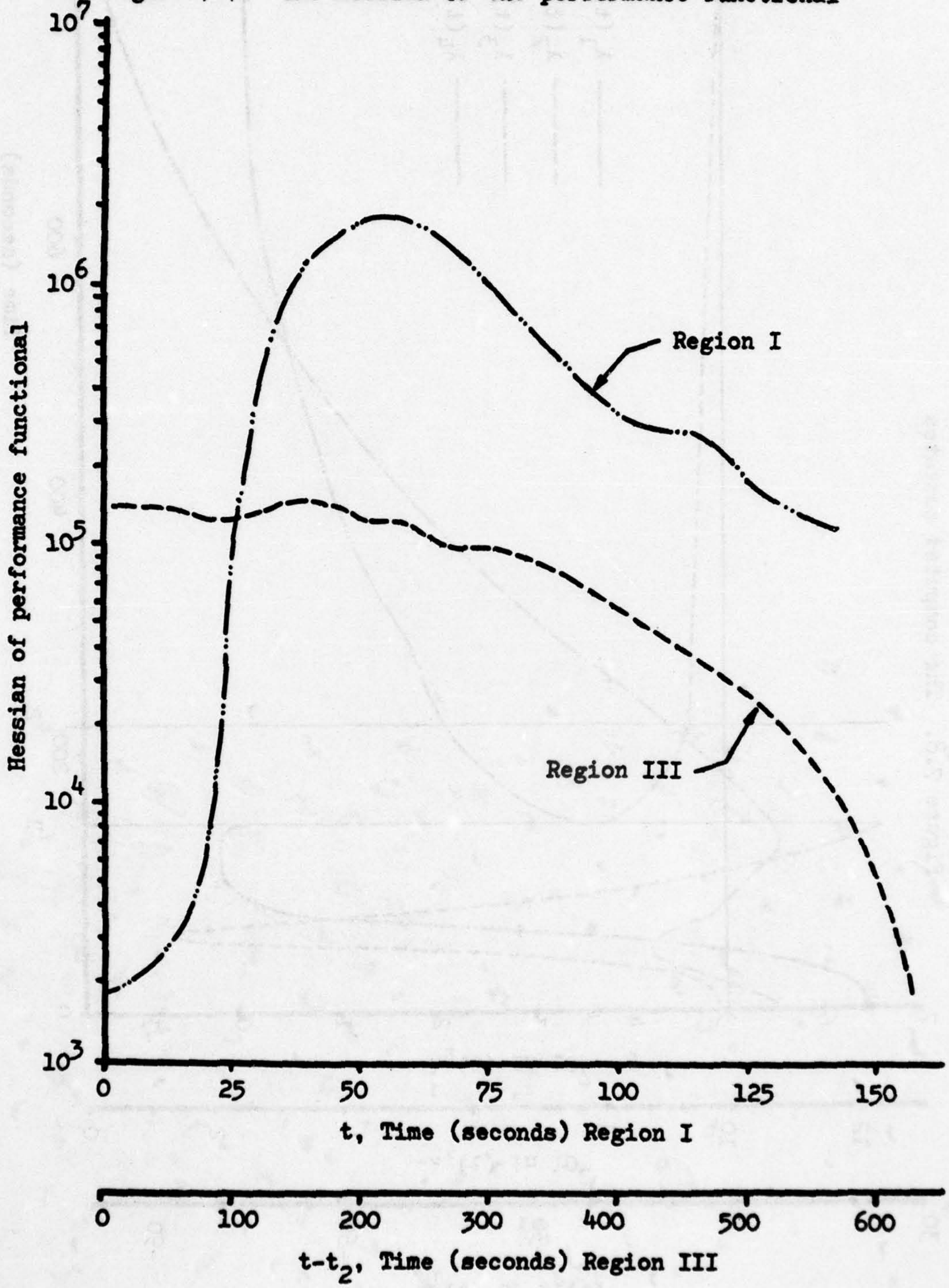


Figure 7.8. The computed costates

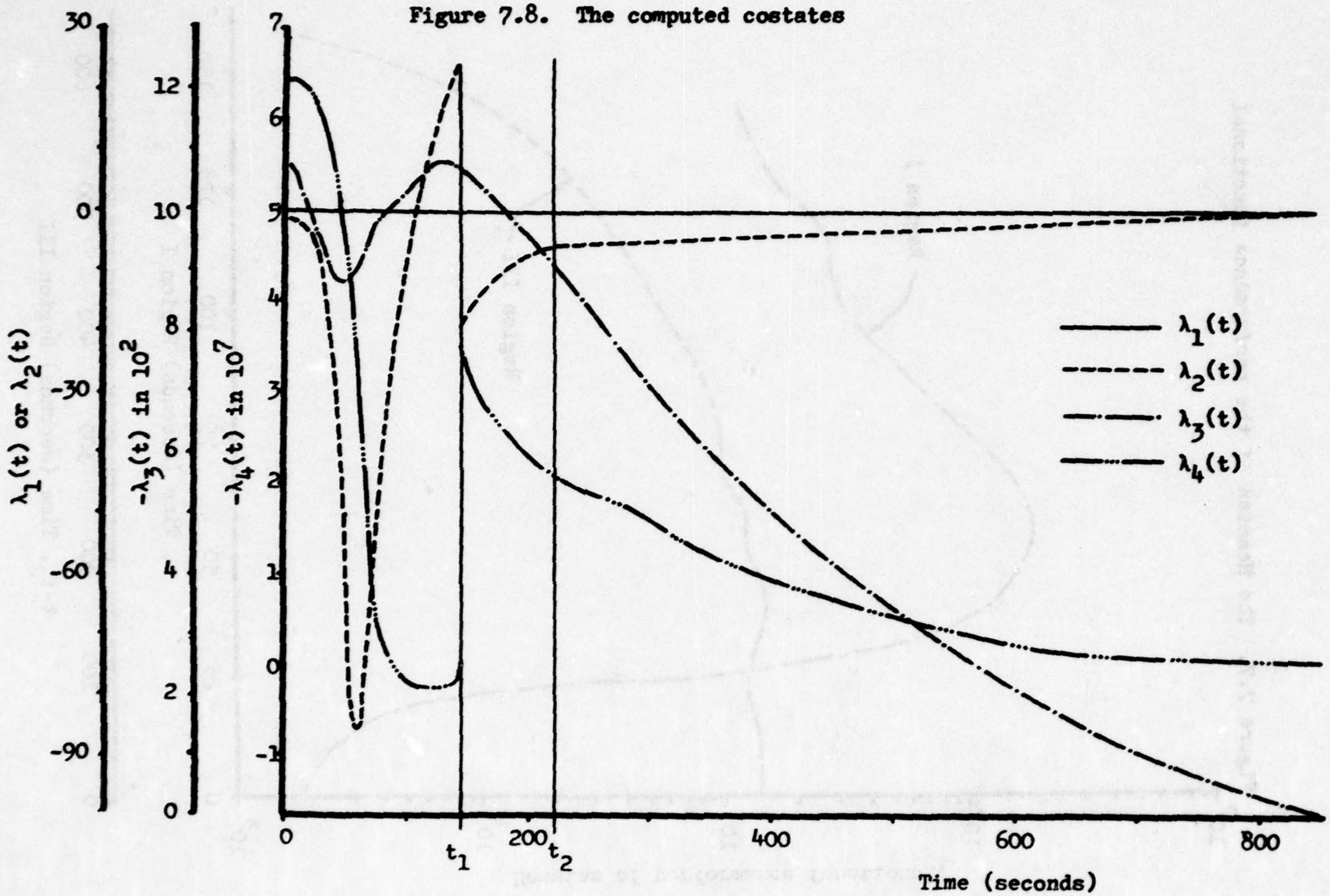


TABLE 1

The dynamics of the maximum range problem after 10 iterations, region I

Time (seconds)	Range (miles)	Altitude (10^3 feet)	Velocity (10^3 ft/sec)	Flight Path Angle (degrees)
0	0.0	340.00	28.000	-8.0214
5	22.458	320.53	28.020	-7.9626
10	44.954	301.19	28.037	-7.9006
15	67.487	281.99	28.050	-7.8319
20	90.049	262.98	28.051	-7.7491
25	112.63	244.20	28.031	-7.6365
30	135.21	225.79	27.967	-7.4622
35	157.73	207.99	27.818	-7.1647
40	180.12	191.30	27.520	-6.6389
45	202.23	176.50	26.994	-5.7397
50	223.85	164.69	26.191	-4.3533
55	244.78	156.89	25.175	-2.5466
60	264.89	153.47	24.127	- .64277
65	284.21	153.86	23.227	.96062
70	302.86	156.95	22.544	2.0435
75	321.04	161.54	22.053	2.6041
80	338.86	166.70	21.699	2.7499
85	356.43	171.78	21.428	2.6317
90	373.79	176.51	21.212	2.4488
95	391.00	180.83	21.034	2.2374
100	408.07	184.72	20.885	2.0100
105	425.03	188.15	20.757	1.7740
110	441.90	191.14	20.647	1.5338
115	458.68	193.68	20.548	1.2920
120	475.38	195.78	20.460	1.0500
125	492.01	197.43	20.379	.80862
130	508.58	198.65	20.304	.56854
135	525.09	199.45	20.233	.33703
140	541.55	199.86	20.167	.13610
144.4	555.98	199.96	20.111	.00001

TABLE 2

The dynamics of the maximum range problem after 10 iterations, region II

Time [($t-t_1$) seconds]*	Range (miles)	Velocity (10^3 ft/sec)
0	555.98	20.111
5	572.33	20.025
10	588.60	19.938
15	604.81	19.852
20	620.95	19.765
25	637.01	19.677
30	653.00	19.590
35	668.92	19.501
40	684.77	19.413
45	700.55	19.323
50	716.25	19.233
55	731.88	19.143
60	747.44	19.051
65	762.92	18.959
70	778.33	18.866
75	793.66	18.773
79.47	806.40	18.693

*The entering time $t_1 = 144.4$.

TABLE 3

The dynamcis of the maximum range problem after 10 iterations, region III

Time $(t-t_2)$* (seconds)	Range $(R-R_2)$** (miles)	Altitude (10^3 feet)	Velocity (10^3 ft/sec)	Flight Path Angle (degrees)
0	0.0	200.00	18.693	0.0
50	148.50	198.90	17.776	- 0.20551
100	289.55	192.94	16.828	- 0.56930
150	422.13	183.95	15.663	- 0.62105
200	544.26	177.35	14.277	- 0.38409
250	654.91	172.87	12.865	- 0.45921
300	754.10	165.86	11.446	- 0.88688
350	841.25	156.10	9.8873	- 1.1610
400	915.13	146.37	8.2082	- 1.3492
450	975.31	135.78	6.5411	- 2.1070
500	1022.1	121.74	4.9259	- 3.7584
550	1055.9	103.47	3.3970	- 6.8382
600	1077.6	82.420	2.0024	-12.231
628.75	1085.1	70.000	1.3706	-19.150

* Exit time $t_2 = 79.47 + 144.4$

** Range at exit time $R_2 = 806.40$ miles

SECTION VIII

CONCLUSIONS AND REMARKS

From the computational experience based on the problems studied above, the method of conjugate gradients has been shown to be a useful computational tool in solving both linear and nonlinear optimal control problems with state variable constraint. The method is basically simple and relatively easy to program. Although the search directions are only locally conjugate with respect to the second Frechet derivative of the performance functional, they still provide satisfactory convergence. The results presented in Section VI indicate that the conjugate gradient method has a higher rate of convergence in comparison with the method of steepest descent, but the difference in the rate of convergence is less pronounced for this constraint problem, as compared with the cases of unconstrained problems reported by other investigators [13], [14], because of the following reasons: (i) The set of admissible controls Q is restricted, and consequently only small step size in the search direction is permitted in Region I. That is, the convergence is along the expanding sequence of sets $\{B_n \cap Q\}$ instead of expanding sequence of subspaces. (ii) The rate of convergence in Region III depends heavily on the choice of the exit corner in each iteration. A considerable portion of computational time in each iteration is devoted to the determination of the optimum step size in the search (although the exact optimum is not essential) and the determination of jumps in the costate at the entering corner.

To assure that the sequence of approximating controls converges to optimum, it suffices to have the initial estimated control so that the second Frechet derivative evaluated there is positive definite. The method of conjugate gradients, like many other optimization techniques, cannot differentiate local minimum from absolute minimum, and consequently the initial estimated control must be selected cautiously unless the given problem is known to have only one minimum. In converting the constraint control problem to an equivalent unconstrained one by introducing a penalty function, the computational process involves more time in contrast to the approach which considers the constraints directly, but it requires less programming work. Its effectiveness depends heavily on the proper choice of the function π. We have treated only the control systems that are time-invarying, but the extension of the conjugate gradient method to encompass the time-varying systems is straightforward.

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*U.S.Government Printing Office: 1979 — 657-002/664